Club Networks with Multiple Memberships and Noncooperative Stability

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Abstract

Modeling club structures as bipartite directed networks, we formulate the problem of club formation as a noncooperative game of network formation and we identify conditions on network formation rules and players' network payoffs sufficient to guarantee the existence of a unique nonempty set of Nash club networks stable (externally and internally) with respect to noncooperative path dominance. Our sufficient conditions on network formation rules require that each player be able to *move freely and unilaterally* from one club to another and *choose freely and unilaterally* feasible activities within those clubs joined by the player. We refer to our conditions on rules as noncooperative free mobility. Our sufficient conditions on network payoffs require that players' payoffs be additively separable in player-specific payoffs and externalities and that payoff externalities, a function of club membership, club activities, and crowding, be identical across players. We refer to our conditions on payoffs as *additive separability and externality homogeneity.* We then show that under noncooperative free mobility, additive separability, and externality homogeneity, the noncooperative game of club network formation is a potential game over directed club networks. The existence of a unique nonempty set of Nash club networks, stable with respect to noncooperative path dominance, then follows easily.

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1 Introduction

Club theory and the theory of local public good provision has a long history in economics, dating back to seminal papers of Charles Tiebout (1956) and James Buchanan (1965). Three types of approaches have been applied: price taking equilibrium theory; cooperative game theory, and; non-cooperative models of club/jurisdiction formation. There has been very little study, however, of club models where players can belong to multiple clubs, a situation introduced in Shubik and Wooders (1982).[1] Also, even in situations allowing multiple memberships in clubs, no account is taken of the fact that individuals may be connected in different ways to the same club and have different connections with different clubs. Networks appear to provide a promising approach to modeling strategic club formation where players can have multiple club memberships with different connections within clubs and across clubs.

Modeling club structures as bipartite directed networks, we formulate the problem of club formation as a noncooperative game of network formation and identify conditions on network formation rules and players' network payoffs sufficient to guarantee the existence of a unique nonempty set of Nash club networks that are stable (externally and internally) with respect to noncooperative path dominance, introduced in Page and Wooders (2005). Our sufficient conditions on network formation rules require that each player be able to move freely and unilaterally from one club to another and choose freely and unilaterally feasible activities within those clubs joined by the player. We refer to our conditions on rules as *noncooperative free mobility.* Our sufficient conditions on network payoffs require that players' payoffs be additively separable in player-specific payoffs and externalities and that payoff externalities – a function of club membership, club activities, and crowding – be identical across players. We refer to our conditions on payoffs as *additive separability* and *externality homogeneity.* We then show that under noncooperative free mobility, additive separability, and externality homogeneity, the noncooperative game of club network formation is a potential game over directed club networks. The existence of a unique nonempty set of Nash club networks, stable with respect to noncooperative path dominance, then follows easily.

This paper grew out of earlier work (Page and Wooders, 2005) where we develop a game theoretic model of network formation whose primitives consist of a feasible set of networks (bipartite or otherwise), player preferences, the rules of network formation, and a dominance relation. A specification of the primitives induces an abstract game consisting of (i) a feasible set of networks and (ii) a path dominance relation defined on the feasible set of networks. Under the path dominance relation, a network G path dominates another network G' if there is a finite sequence of networks, beginning with G and ending with G' where each network along the sequence dominates its predecessor.[2] Using this induced abstract game as our basic analytic

[1] Shubik and Wooders (1982) is the first contribution known to us allowing multiple memberships. Allouch an Wooders (2007) presents a discussion of this literature, primarily concerned with nonemptiness of cores, existence of equilibrium, and core-equilibrium equivalence.

[2] Stated formally, given feasible set of networks $\mathbb{G}$ and dominance relation $>$, network $G' \in \mathbb{G}$ (weakly) path dominates network $G \in \mathbb{G}$, written $G' \geq_p G$, if $G' = G$ *or* if there exists a *finite*

tool we demonstrate that for any set of primitives the following results hold:

1. The feasible set of networks contains a unique, finite, disjoint collection of nonempty subsets each constituting a *strategic basin of attraction.* Given preferences and the rules of governing network formation, these basins of attraction are the absorbing sets of the process of network formation modeled via the game.

2. A stable set (in the sense of von Neumann Morgenstern) with respect to path dominance consists of one network from each basin of attraction.

3. The path dominance core, defined as a set of networks having the property that no network in the set is path dominated by any other feasible network, consists of one network from each basin of attraction containing a *single* network. Thus, the path dominance core is contained in each stable set and is nonempty if and only if there is a basin of attraction containing a single network.[3]

If the rules of network formation are noncooperative (as in this paper), then the set of networks contained in the path dominance core is equal to the set of Nash networks, and thus the set of Nash networks is nonempty if and only if there is a basin of attraction containing a single network. Viewed in this light, we show here that in games of club network formation satisfying noncooperative free mobility, additive separability, and externality homogeneity all basins of attraction contain a single network. In fact, we accomplish this by showing that any game of club network formation satisfying our three conditions is a potential game (see Monderer and Shapley, 1996).

Our research is also related to that of Kalai and Schmeidler (1977) since in any network formation game the union of basins of attraction is equal to the admissible set, introduced in their work.[4] To define the admissible set, take as given a set of feasible alternatives, denoted by S, a dominance relation M and the transitive closure of M, denoted by $\widehat{M}$. The *admissible set* is the set $A(S,M) = \{x \in S \ : \ y \in S$ and $y\widehat{M}x$ imply $x\widehat{M}y\}$. The admissible set describes those outcomes that are likely to be reached by any dynamic process that respects preferences. The admissible set concept can be applied to a host of game-theoretic situations, ranging from non-cooperative games, where a coalition consists of an individual player, to fully cooperative games, where any coalition can be allowed to form. As shown by Kalai and Schmeidler through a series of examples, the relationship of the admissible set to the set of Nash equilibrium depends on the definition of the dominance relation and, in some cases,

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sequence of networks $\{G_k\}_{k=0}^n$ in $\mathbb{G}$ with $G = G_0$ and $G' = G_n$ such that for $k = 1, 2, \ldots, n$

$$G_k > G_{k-1}.$$

The path dominance relation $\geq_p$ induced by the dominance relation $>$ is sometimes referred to as the transitive closure of $>$.

[3] Put differently, the path dominance core is empty if and only if *all* basins of attraction contain multiple networks.

[4] See also Kalai, Pazner and Schmeidler (1976) and Shenoy (1980).

the set of Nash equilibrium and the admissible set coincide. It is interesting to note that if the dominance relation is defined based on a notion of "possible replies", which can be thought of as "improving replies" (rather than best replies in the usual sense), then the admissible set is equivalent to the set of Nash equilibrium. In the framework of the current paper, in part because of the finiteness of the strategy sets, each Nash equilibrium strategy profile is a basin of attraction and the union of all basins of attraction coincides with (the network rendition of) the admissible set.

2 Club Networks with Multiple Memberships

We begin by introducing the notion of a club network where players can have multiple club memberships. Using bipartite networks we are able to represent any such club structure in a compact and precise way.

Let D be a finite set of players consisting of two or more players with typical element denoted by d and let C be a finite set of club types (or alternatively, a set of club labels or club locations) with typical element denoted by c. Finally, let A be a finite set of arcs (or actions) potentially available to all players. For each player d and club c, denote by $A(d, c)$ the feasible set of actions that can be taken by player d in club c.

Definition 1 (Club Networks with Multiple Memberships)

Given a finite set of players D, a finite set of clubs C, and a finite set of arcs A, a club network G is a nonempty subset of $A \times (D \times C)$ such that (i) for all players $d \in D$, the section of G at d given by

$$G(d) := \{(a, c) \in A \times C : (a, (d, c)) \in G\} \tag{1}$$

is nonempty; and (ii) for all $(a, (d, c)) \in G$, $a \in A(d, c)$. Let $\mathbb{K}$ denote the collection of all such club networks. ■

Given network $G \in \mathbb{K}$, $(a, (d, c)) \in G$ means that player d is a member of club c and takes action $a \in A(d, c)$ in club c. The section of G at d is the set of action-club pairs summarizing the clubs to which player d belongs and the action taken by player d in each of those clubs. The set

$$G(a, c) := \{d \in D : (a, (d, c)) \in G\ \} \tag{2}$$

(i.e., the section of G at (a, c)) is the set of all players who, in club network $G \in \mathbb{K}$, are members of club c and take action a in club c. Thus, the cardinality of the set $G(a, c)$, denoted by $|G(a, c)|$, is the total number of players who are members in club c and take action a in club c, and the sum

$$\sum_{a \in A} |G(a, c)|$$

is the total number of players active in club c.[5]

[5] If $G(a, c) = \emptyset$, then $|G(a, c)| = 0$.

Example 1 (Marketing Networks as Club Networks with Multiple Memberships) Suppose there are five firms $D = \{d_1, d_2, d_3, d_4, d_5\}$, two markets $C = \{c_1, c_2\}$, where c_1 = New York and c_2 = Paris, and three possible product lines $A = \{a_1, a_2, a_3\}$. Each firms feasible product lines appear in the list below:

$$A(d_1, c) = \{a_1, a_3\} \text{ for all } c \in C,$$

$$A(d_2, c) = \{a_1, a_2\} \text{ for all } c \in C,$$

$$A(d_3, c) = \{a_2, a_3\} \text{ for all } c \in C,$$

$$A(d_4, c) = \{a_2, a_3\} \text{ for all } c \in C,$$

$$A(d_5, c) = \{a_1, a_3\} \text{ for all } c \in C.$$

Marketing network G_0 depicted in Figure 1 represents one possible product line-market profile for firms D.

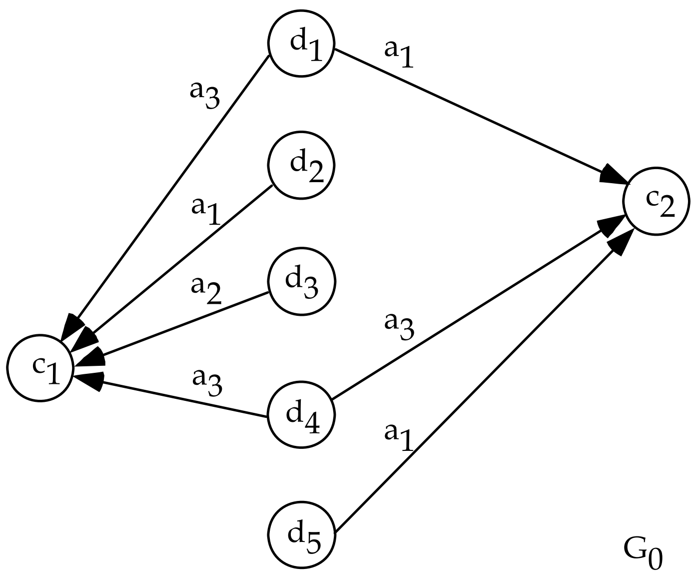

Figure 1: Marketing Network G_0

In marketing network G_0 both firms d_1 and d_5 offer product line a_1 in the Paris market (i.e., in the c_2 market), while no firm offers product line a_2 in the Paris market, and only one firm, d_4, offers product line a_3 in the Paris market. Thus,

$$G_0(a_1, c_2) = \{d_1, d_5\}, \quad G_0(a_2, c_2) = \emptyset, \quad \text{and } G_0(a_3, c_2) = \{d_4\}.$$

Also, note that in marketing network G_0 two firms, d_1 and d_4, are active in the New York and Paris markets. Firms d_2 and d_3 specialize in the New York market, while firm d_5 specializes in the Paris market. Thus,

$$G_0(d_1) = \{(a_3, c_1), (a_1, c_2)\} \quad G_0(d_4) = \{(a_3, c_1), (a_3, c_2)\}$$

$$G_0(d_2) = \{(a_1, c_1)\} \quad G_0(d_3) = \{(a_2, c_1)\}$$

$$G_0(d_5) = \{(a_1, c_2)\} .$$

Note that all product line offerings are feasible (see the list above). Finally, note that four firms are active in the New York market; that is

$$G_0(c_1) := \cup_{a \in A} G_0(a, c_1) = \{d_1, d_2, d_3, d_4\} .$$

■

We shall maintain the following assumption throughout:

(A-1) (noncooperative free mobility) Each player can move freely and unilaterally from one club to another and each player can choose freely and unilaterally his feasible activity within the club. This means that a player can drop his membership and activity in any given club and join any other club and take any other feasible action without bargaining with or seeking the permission of any player or group of players. In this sense, our model of club formation, as a game over club networks with moral hazard and multiple memberships, is noncooperative. ■

The assumption of noncooperative free mobility is quite common in other models of noncooperative network formation (see, for example, Bala and Goyal 2000). Under the assumption of noncooperative free mobility, each player can alter any existing club network by simply switching his memberships and/or changing his activities.

Example 2 Figure 2 depicts the marketing network which results when firm d_1 noncooperatively changes its product line-market profile from

$$G_0(d_1) = \{(a_3, c_1), (a_1, c_2)\} \quad \text{to} \quad G_1(d_1) = \{(a_3, c_1), (a_1, c_2), (a_3, c_1)\} .$$

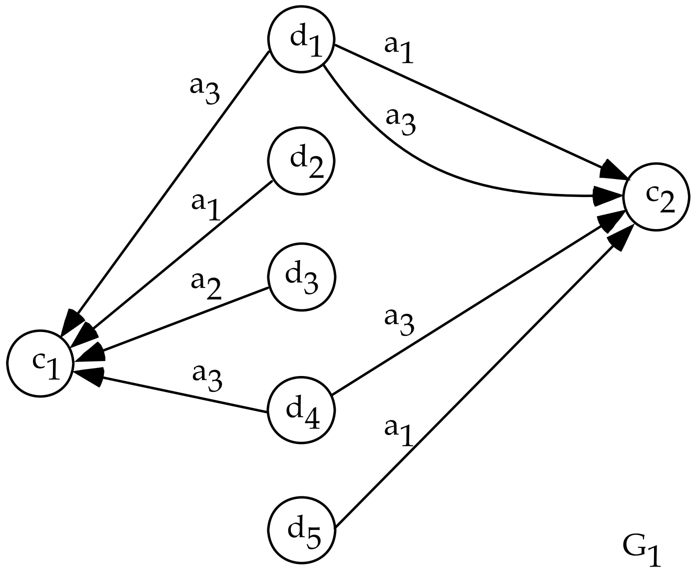

Figure 2: Marketing Network G_1

This change is brought about by firm d_1 adding product line a_3 to its offerings in the Paris market. Note that in marketing network G_1 product line a_1 is offered by two firms, d_1 and d_5, and product line a_3 is offered by firms d_1 and d_4. ■

The noncooperative move by firm d_1 changing marketing network G_0 in Example 1 to marketing network G_1 above is depicted in Figure 3 below.

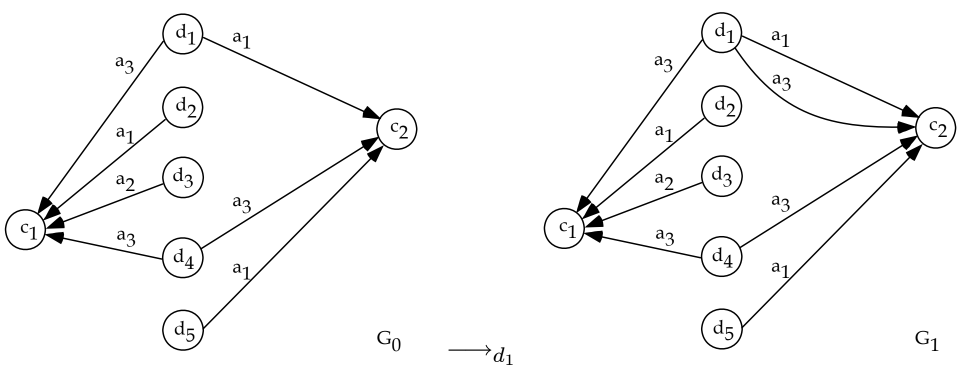

Figure 3: The noncooperative move by firm d_1 from network G_0 to network G_1

The move from G_0 to G_1 brought about by firm d_1 is denoted by

$$G_0 \rightarrow_{d_1} G_1,$$

where the resulting network G_1 is given by

$$G_1 = G_0 \backslash (d_1 \times G_0(d_1)) \cup (d_1 \times G_1(d_1)),$$

and where

$$d_1 \times G_0(d_1) := \{(a, (d_1, c)) : (a, c)) \in G_0(d_1)\}$$
$$\text{and}$$
$$d_1 \times G_1(d_1) := \{(a, (d_1, c)) : (a, c)) \in G_1(d_1)\}.$$

Referring to Figure 3, note that

$$d_1 \times G_0(d_1) := \{(a_3, (d_1, c_1)), (a_1, (d_1, c_2))\}$$
and
$$d_1 \times G_1(d_1) := \{(a_3, (d_1, c_1)), (a_1, (d_1, c_2)), (a_3, (d_1, c_2))\}.$$

3 Payoffs, Potentials, and Nash Club Networks

3.1 Payoffs

We will assume that (i) each players payoffs are additively separable in player specific payoffs, internal effects, and external effects; and (ii) that internal effects and external effects are homogenous across players. In particular, we will maintain the following assumption throughout:

(A-2) (additive separability and externality homogeneity) Each player's payoff over club networks is given by

$$v_d(G) = \sum_{(a,c)\in G(d)} r_d(a,c) + \sum_{(a,c)\in G(d)} I_{(a,c)}(|G(a,c)|) + \sum_{(a,c)\in G(d)^c} E_{(a,c)}(|G(a,c)|), \tag{3}$$

where,

$G(d)^c$ is the complement of the set $G(d)$ in $A \times C$,

$r_d(a,c)$ is the player-specific payoff generated by the action-club pair, $(a,c) \in G(d)$, chosen by player d in network G,

$I_{(a,c)}(|G(a,c)|)$ is the payoff externality generated by the number of players who choose the action-club pair, (a,c), chosen by player d in network G,

$E_{(a,c)}(|G(a,c))|)$ is the payoff externality generated by the number of players who choose an action-club pair (a,c) in network G not contained in the set of action-club pairs $G(d)$ chosen by player d in network G, and

$$\sum_{(a,c)\in G(d)} I_{(a,c)}(|G(a,c)|) + \sum_{(a,c)\in G(d)^c} E_{(a,c)}(|G(a,c)|)$$

is the sum of all of these payoff externalities. ■

We will refer to the quantity $I_{(a,c)}(|G(a,c)|)$ as the *internal effect* of action-club pair (a,c) on a player in club network G and we will refer to $E_{(a,c)}(|G(a,c)|)$ as the *external effect* of action-club pair (a,c) on a player in club network G. The internal effect $I_{(a,c)}(|G(a,c)|)$ accrues to a player, say player d, if and only if (a,c) is contained in the set of action-club pairs chosen by player d in network G; that is, if and only if $(a,c) \in G(d)$. The external effect $E_{(a,c)}(|G(a,c)|)$ accrues to a player, say player d, if and only if (a,c) is not contained in the set of action-club pairs chosen by player d in network G; that is, if and only if $(a, c) \notin G(d)$. Note that for each action-club pair (a, c), the functions $I_{(a,c)}(\cdot)$ and $E_{(a,c)}(\cdot)$ are the same for all players. Our specification of player payoffs given in (3) is a network rendition of a specification introduced in Hollard (2000).

3.2 Potentials

Our objective in this section is to show that under the assumptions of noncooperative free mobility (A-1) and payoff separability (A-2), the club network formation game with multiple club memberships is a finite potential game. This will allow us to show, in a manner similar to Monderer and Shapley (1996), that under assumptions (A-1) and (A-2) all noncooperative club network formation games with multiple membership possesses Nash club networks.

We begin with three definitions.

Definition 2 (noncooperative network changes and noncooperative club network formation games)

(1)A noncooperative network change from network $G_0 \in \mathbb{K}$ to network $G_1 \in \mathbb{K}$ is a change brought about by a single player, say player d_1, denoted by $G_0 \to_{d_1} G_1$ such that (i) $G_1 = G_0 \backslash (d_1 \times G_0(d_1)) \cup (d_1 \times G_1(d_1))$, and (ii) $G_0(d_1) \neq G_1(d_1)$.

(2) A noncooperative club network formation game, $(\mathbb{K}, v_d(\cdot))_{d \in D}$, is a game where only noncooperative network changes are allowed. ■

Definition 3 (Nash clubs)

A club network $G_0 \in \mathbb{K}$ is said to be a Nash club network for the noncooperative club network formation game $(\mathbb{K}, v_d(\cdot))_{d \in D}$ if for all noncooperative changes $G_0 \to_{d_1} G_1$, $v_{d_1}(G_0) \geq v_{d_1}(G_1)$. Let $\mathbb{NCN}$ denote the set of all Nash club networks. ■

Definition 4 (potential game)

The noncooperative club network formation game $(\mathbb{K}, v_d(\cdot))_{d \in D}$ is a potential game if there exists a function,

$$P(\cdot) : \mathbb{K} \to \mathbb{R}$$

such that for all for all noncooperative changes $G_0 \to_{d_1} G_1$

$$v_{d_1}(G_1) - v_{d_1}(G_0) = P(G_1) - P(G_0).$$

■

It is easy to see that if $(\mathbb{K}, v_d(\cdot))_{d \in D}$ is a potential game with potential $P(\cdot)$, then any club network contained in $\arg\max_{G \in \mathbb{K}} P(G)$ is a Nash club network for $(\mathbb{K}, v_d(\cdot))_{d \in D}$. Moreover, since $\mathbb{K}$ is finite, $\arg\max_{G \in \mathbb{K}} P(G)$ is nonempty. Thus, one way to resolve the Nash problem for club network formation games is to show that these games possess potential functions. Our next objective, therefore, will be to show that for club network formation games satisfying noncooperative free mobility (A-1) and payoff separability (A-2) a potential function can be constructed.

Following Hollard (2000), let

$$\Phi_{(a,c)}(k) = I_{(a,c)}(k) - E_{(a,c)}(k-1),\ k = 0, 1, \ldots, |D|\,.$$

In club network $G \in \mathbb{K}$, if player d chooses action-club pairs $G(d)$ and $(a,c) \in G(d)$, then

$$\Phi_{(a,c)}(|G(a,c)|) = I_{(a,c)}(|G(a,c)|) - E_{(a,c)}(|G(a,c)| - 1)$$

is the difference between the internal effect derived by player d in network G from being in group $G(a,c)$ and the external effect player d would derive from group $G(a,c)$ if player d were to leave that group by noncooperatively choosing some other action-club (a',c') not equal to (a,c).

Our main result is the following:

Theorem 1 *(Club network formation names with multiple memberships satisfying noncooperative free mobility, additive separability, and externality homogeneity are potential games).*

Let $(\mathbb{K}, v_d(\cdot))_{d\in D}$ be a club network formation game satisfying noncooperative free mobility (A-1), payoff separability, and externality homogeneity (A-2). Then the function

$$P(\cdot) : \mathbb{K} \to \mathbb{R}$$

given by

$$P(G) = \sum_{(a,c)\in A\times C} \left[\sum_{d\in G(a,c)} r_d(a,c) + \sum_{k=0}^{|G(a,c)|} \Phi_{(a,c)}(k) \right], \tag{4}$$

is a potential function for this game.

Since the proof consists primarily of long and tedious elementary algebra it is relegated to the appendix.

3.3 Nash Club Networks

We now have our main result on the existence of Nash club networks for noncooperative club network formation games with multiple memberships.

Theorem 2 *(Club network formation names with multiple memberships satisfying noncooperative free mobility, additive separability, and externality homogeneity possess Nash club networks).*

Let $(\mathbb{K}, v_d(\cdot))_{d\in D}$ be a club network formation game satisfying noncooperative free mobility (A-1), payoff separability, and externality homogeneity (A-2). Then there exists a Nash club network $G^* \in \mathbb{K}$; that is, there exists a club network $G^* \in \mathbb{K}$ such that for all noncooperative changes $G_0 \to_{d_1} G_1$, $G_1 \in \mathbb{K}$, brought about by some player d_1,

$$v_{d_1}(G^*) \geq v_{d_1}(G_1).$$

PROOF: By Theorem 1 the club network formation game, $(\mathbb{K}, v_d(\cdot))_{d\in D}$, is a potential game with potential

$$P(G) = \sum_{(a,c)\in A\times C} \left[\sum_{d\in G(a,c)} r_d(a,c) + \sum_{k=0}^{|G(a,c)|} \Phi_{(a,c)}(k) \right].$$

Since the set of club networks $\mathbb{K}$ is finite, $\arg\max_{G\in\mathbb{K}} P(G)$ is nonempty. Let $G^* \in \arg\max_{G\in\mathbb{K}} P(G)$. Then G^* is a Nash club network. If not, then there exists a noncooperative change in club network G^*, say $G^* \rightarrow_{d_1} G_1 \in \mathbb{K}$, which can be brought about by some player d_1 such that $v_{d_1}(G_1) > v_{d_1}(G^*)$. Because $P(\cdot)$ is a potential,

$$P(G_1) - P(G^*) = v_{d_1}(G_1) - v_{d_1}(G^*) > 0.$$

But this contradicts the fact that $G^* \in \arg\max_{G\in\mathbb{K}} P(G)$. ■

4 The Noncooperative Path Dominance Core

Because all club network formation games satisfying noncooperative free mobility (A-1) and additive separability and externality homogeneity (A-2) are a potential games, much more can be said about stability with respect to noncooperative network changes. In particular, we can show that no noncooperative improvement path forms a circuit and that each club network in $\mathbb{K}$ is either a Nash club network or is a network on a finite, noncooperative improvement path leading to a Nash club network. Thus all club network formation games satisfying (A-1) and (A-2) have nonempty noncooperative path dominance cores (see Page and Wooders, 2005). Now we provide the details.

4.1 Noncooperative Path Dominance

We begin with a definition of noncooperative direct dominance. The definitions and some of the results of this section are special cases or applications from Page and Wooders (2005). Since we treat potential games, however, we are able to obtain stronger results.

Definition 5 (Noncooperative Direct Dominance)

(1) We say that club network $G' \in \mathbb{K}$ noncooperatively directly dominates club network $G \in \mathbb{K}$ *via player* d, written $G' \rhd_d G$, if for player $d \in D$,

$$G \rightarrow_d G' \text{ and } v_d(G') > v_d(G).$$

(2) We say that club network $G' \in \mathbb{K}$ noncooperatively directly dominates club network $G \in \mathbb{K}$, written $G' \rhd G$, if for *some* player $d \in D$,

$$G \rightarrow_d G' \text{ and } v_d(G') > v_d(G).$$

■

Thus, G' noncooperatively directly dominates G if there is *some* player d *who can* noncooperatively change club network G to club network G' by changing his action-club profile and *who wants* to change club network G to club network G'.[6]

[6] Written compactly,

$$G' \rhd G \Leftrightarrow \exists d \in D \text{ such that } G' \rhd_d G.$$

The noncooperative direct dominance relation $\rhd$ on $\mathbb{K}$ induces a noncooperative path dominance relation $\geq_{np}$ on $\mathbb{K}$ specified as follows:

Definition 6 (Noncooperative Path dominance)

Club network $G' \in \mathbb{K}$ (weakly) noncooperatively path dominates club network $G \in \mathbb{K}$ with respect to $\rhd$, written $G' \geq_{np} G$, if $G' = G$ or if there exists a *finite* sequence of club networks $\{G_k\}_{k=0}^{h}$ in $\mathbb{K}$ with $G_h = G'$ and $G_0 = G$ such that for $k = 1, 2, \ldots, h$

$$G_k \rhd G_{k-1}.$$

We refer to such a finite sequence of club networks as a *finite improvement path* and we say network G' is $\rhd$-*reachable* from network G if there exists a finite improvement path from G to G'. Thus,

$$G' \geq_{np} G \text{ if and only if } \begin{cases} G' \text{ is } \rhd\text{-reachable from } G, \text{ or} \\ G' = G. \end{cases} \tag{5}$$

■

Note that if club network G' is $\rhd$-reachable from G via some finite sequence of networks $\{G_k\}_{k=0}^{h}$ in $\mathbb{K}$ (not necessarily unique), then corresponding to this sequence there is a unique finite sequence of players $\{d_k\}_{k=1}^{h} \subseteq D$ such that $G_k \rhd_{d_k} G_{k-1}$.

4.2 Circuits, Equivalence, and Isolation

If club network G_1 is reachable from club network G_0, and if G_0 is reachable from G_1 (i.e., if $G_1 \geq_{np} G_0$ *and* $G_0 \geq_{np} G_1$), then networks G_1 and G_0 are said to be noncooperatively equivalent, denoted by $G_1 \equiv_{np} G_0$. If networks G_0 and G_1 are noncooperatively equivalent but not equal (i.e., $G_0 \neq G_1$), then there is a noncooperative domination path leading from network G_0 back to network G_0 which includes network G_1. We call such a path a *noncooperative circuit*, and we say that networks G_0 and G_1 are on the same circuit. Thus, if networks G_0 and G_1 are equivalent then either networks G_1 and G_0 coincide ($G_0 = G_1$) or G_1 and G_0 are on the same circuit. Finally, if network G is *not* reachable from any network in $\mathbb{K}$ and if no network in $\mathbb{K}$ is reachable from G, then network G is *noncooperatively isolated* (i.e., network $G \in \mathbb{K}$ is noncooperatively isolated if there does not exist a network $G' \in \mathbb{K}$ with $G' \geq_{np} G$ or $G \geq_{np} G'$).

4.3 The Noncooperative Path Dominance Game of Club Network Formation

Corresponding to the noncooperative game $(\mathbb{K}, v_d(\cdot))_{d \in D}$ of club network formation there is an abstract game (in the sense of vonNewmann-Morgenstern) of club network formation with respect to noncooperative path dominance given by the pair

$$(\mathbb{K}, \geq_{np}).$$

Definition 7 (The Noncooperative Path Dominance Core)

A club network $G \in \mathbb{K}$ is said to be a core network of $(\mathbb{K}, \geq_{np})$ if there does not exist a network $G' \in \mathbb{K}$, $G' \neq G$, such that $G' \geq_{np} G$. Let $\mathbb{C}$ denote the set of all core networks and call $\mathbb{C}$ the noncooperative path dominance core of $(\mathbb{K}, \geq_{np})$.

In attempting to identify those club networks which are in the noncooperative path dominance core, club networks *without descendants* are of particular interest.

4.4 Club Networks Without Descendants

If club network G_1 noncooperatively path dominates club network G_0, so that $G_1 \geq_{np} G_0$ but G_1 and G_0 are not equivalent (i.e., not $G_1 \equiv_{np} G_0$), then network G_1 is a *descendant* of network G_0 and we write

$$G_1 >_{np} G_0. \tag{6}$$

Thus, if G_1 is a descendant of G_0, then there is a noncooperative domination path from G_0 to G_1, but there is not a noncooperative domination path from G_1 back to G_0.

Definition 8 (Club Networks Without Descendants)

We say that club network $G' \in \mathbb{K}$ *has no descendants in* $\mathbb{K}$ if for any network $G \in \mathbb{K}$

$$G \geq_{np} G' \text{ implies that } G \equiv_{np} G'.$$

Thus, if G' has no descendants then $G \geq_{np} G'$ implies that G and G' coincide or lie on the same circuit.[7]

Here is our main result concerning club networks without descendants. The following Theorem is a rendition of Page and Wooders (2005, Theorem 1) for club network formation games**.**

Theorem 3 (*All noncooperative path dominance games of club network formation have networks without descendants)*

Let $(\mathbb{K}, \geq_{np})$ *be a noncooperative path dominance games of club network formation. For every club network* $G \in \mathbb{K}$ *there exists a network* $G' \in \mathbb{K}$ *such that* $G' \geq_{np} G$ *and* G' *has no descendants.*

Proof. Let G_0 be any club network in $\mathbb{K}$. If G_0 has no descendants then we are done. If not choose G_1 such that $G_1 >_{np} G_0$. If G_1 has no descendants then we are done. If not, continue by choosing $G_2 >_p G_1$. Proceeding iteratively, we can generate a sequence, $G_0, G_1, G_2, \ldots$. Now observe that in a finite number of iterations we must come to a club network $G_{k'}$ without descendants. Otherwise, we could generate an infinite sequence, $\{G_k\}_k$ such that for all k,

$$G_k >_p G_{k-1}.$$

However, because $\mathbb{K}$ is finite this sequence would contain at least one club network, say $G_{k'}$, which is repeated an infinite number of times. Thus, all the networks in

[7] Note that any isolated network is by definition a network without descendants.

the sequence lying between any two consecutive repetitions of $G_{k'}$ would be on the same circuit, contradicting the fact that for all k, G_k is a descendant of G_{k-1} (i.e., $G_k >_p G_{k-1}$). ■

While Theorem 3 does not require assumption (A-1) and/or assumption (A-2), our next Theorem requires both (A-1) and (A-2).

Theorem 4 *(Non-cooperative path dominance games of club network formation satisfying (A-1) and (A-2) have no circuits)*

Let $(\mathbb{K}, v_d(\cdot))_{d\in D}$ *be a noncooperative club network formation game satisfying non-cooperative free mobility (A-1), payoff separability, and externality homogeneity (A-2) with corresponding noncooperative path dominance game* $(\mathbb{K}, \geq_{np})$. *Then the following statements are true.*

(1) $\mathbb{K}$ *contains no noncooperative circuits.*

(2) For all club networks $G' \in \mathbb{K}$ *without descendants, if* $G \in \mathbb{K}$ *noncooperatively path dominates* G'*, then club networks* G *and* G' *are equal.*

Thus, if G' has no descendants, then $G \geq_{np} G'$ implies that G and G' are equal. Theorem 3 is an immediate consequence of the fact that all noncooperative club network formation games satisfying (A-1) and (A-2) are potential games.

Let $\mathbb{Z}$ denote the set of all club networks in $\mathbb{K}$ without descendance. By Theorem 3, $\mathbb{Z}$ is nonempty. By Theorem 4, under the assumptions of noncooperative free mobility (A-1) and payoff separability and externality homogeneity (A-2), each club network G contained in $\mathbb{Z}$ is unique in the sense that there are no other networks equivalent to G; that is for each $G \in \mathbb{Z}$,

$$\left\{G' \in \mathbb{K} : G' \equiv_{np} G\right\} = \{G\}.$$

Given these observations, we can state the following:

Theorem 5

Let $(\mathbb{K}, v_d(\cdot))_{d\in D}$ *be a noncooperative club network formation game satisfying non-cooperative free mobility (A-1), payoff separability, and externality homogeneity (A-2) with corresponding noncooperative path dominance game* $(\mathbb{K}, \geq_{np})$. *Then the following statements are true.*

(1) The noncooperative path dominance core is equal to the set of club networks without descendance; that is,

$$\mathbb{C} = \mathbb{Z}.$$

(2) The set of Nash club networks is equal to the set of club networks without descendance; that is,

$$\mathbb{NCN} = \mathbb{Z}.$$

By Theorem 5, under assumptions (A-1) and (A-2),

$$\mathbb{NCN} = \mathbb{C} = \mathbb{Z}.$$

Moreover, under assumptions (A-1) and (A-2), the noncooperative game of club network formation $(\mathbb{K}, v_d(\cdot))_{d\in D}$ is a potential game with potential function

$$P(G) = \sum_{(a,c)\in A\times C} \left[\sum_{d\in G(a,c)} r_d(a,c) + \sum_{k=0}^{|G(a,c)|} \Phi_{(a,c)}(k) \right].$$

Thus,

$$\arg\max_{G\in\mathbb{K}} P(G) \subseteq \mathbb{NCN} = \mathbb{C} = \mathbb{Z}.$$

5 Some relationships to the literature

Relative to the applications of the admissible set concept in Kalai and Schmeidler (1977), our model is restrictive in that both the number of arcs and nodes are finite. Much of the depth and beauty in the Kalai-Schmeidler results is in their treatment of situations with continuous strategy spaces and payoff sets. Fortunately, the Kalai-Schmeidler methods can all be applied to infinite networks, as we demonstrate in research in progress.

The literature on economies with local public goods or clubs most closely related to the current paper is the line of literature including, for example, Demange (1994, 2005) and Konishi, Le Breton and Weber (1997, 1998), who study economies with a fixed number of jurisdictions and free mobility of agents between jurisdictions. A club model in which players could belong to multiple clubs, as in our paper, was introduced in Shubik and Wooders (1982). Unlike our model, however, free entry was not allowed in the Shubik-Wooders models or in subsequent models with multiple memberships.[8]

In the literature on potential games, as we have already noted our results are related to those of Hollard (2000). Important references in this literature include Monderer and Shapley (1996) and Rosenthal (1973).

6 Appendix

Theorem 1 *(Club Network Formation Games with Multiple Memberships are Potential Games).*

Let $(\mathbb{K}, v_d(\cdot))_{d\in D}$ be a club network formation game satisfying noncooperative free mobility (A-1), payoff separability, and externality homogeneity (A-2). Then the function $P(\cdot) : \mathbb{K} \to \mathbb{R}$ given by

$$P(G) = \sum_{(a,c)\in A\times C} \left[\sum_{d\in G(a,c)} r_d(a,c) + \sum_{k=0}^{|G(a,c)|} \Phi_{(a,c)}(k) \right], \tag{7}$$

is a potential function for this game.

[8] See Allouch and Wooders (2007) for a recent discussion of club models with multiple memberships.

PROOF: Let $G_0 \to_{d_1} G_1$ be a noncooperative network change where $G_1 = G_0 \backslash (d_1 \times G_0(d_1)) \cup (d_1 \times G_1(d_1))$ and $G_0(d_1) \neq G_1(d_1)$. We have

$$\left.\begin{aligned} & v_{d_1}(G_1) - v_{d_1}(G_0) \\ & = \left(\sum_{(a,c)\in G_1(d_1)} r_{d_1}(a,c) - \sum_{(a,c)\in G_0(d_1)} r_{d_1}(a,c)\right) \\ & + \left(\sum_{(a,c)\in G_1(d_1)} I_{(a,c)}(|G_1(a,c)|) - \sum_{(a,c)\in G_0(d_1)} I_{(a,c)}(|G_0(a,c)|)\right) \\ & + \left(\sum_{(a,c)\in G_1(d_1)^c} E_{(a,c)}(|G_1(a,c)|) - \sum_{(a,c)\in G_0(d_1)^c} E_{(a,c)}(|G_0(a,c)|)\right). \end{aligned}\right\} \tag{8}$$

First, observe that,

$$\left.\begin{aligned} & \sum_{(a,c)\in G_1(d_1)} r_{d_1}(a,c) - \sum_{(a,c)\in G_0(d_1)} r_{d_1}(a,c) \\ & = \sum_{(a,c)\in G_1(d_1)\backslash G_0(d_1)} r_{d_1}(a,c) - \sum_{(a,c)\in G_0(d_1)\backslash G_1(d_1)} r_{d_1}(a,c). \end{aligned}\right\} \tag{9}$$

Second, observe that

$$\left.\begin{gathered} \text{for all } (a,c) \in (G_1(d_1) \cap G_0(d_1)) \cup (A \times C\backslash(G_1(d_1) \cup G_0(d_1)), \\ |G_1(a,c)| = |G_0(a,c)|. \end{gathered}\right\} \tag{10}$$

Thus,

$$\left.\begin{aligned} & \sum_{(a,c)\in G_1(d_1)} I_{(a,c)}(|G_1(a,c)|) - \sum_{(a,c)\in G_0(d_1)} I_{(a,c)}(|G_0(a,c)|) \\ & = \sum_{(a,c)\in G_1(d_1)\backslash G_0(d_1)} I_{(a,c)}(|G_1(a,c)|) \\ & - \sum_{(a,c)\in G_0(d_1)\backslash G_1(d_1)} I_{(a,c)}(|G_0(a,c)|), \end{aligned}\right\} \tag{11}$$

and

$$\left.\begin{aligned} & \sum_{(a,c)\in G_1(d_1)^c} E_{(a,c)}(|G_1(a,c)|) - \sum_{(a,c)\in G_0(d_1)^c} E_{(a,c)}(|G_0(a,c)|) \\ & = \sum_{(a,c)\in G_0(d_1)\backslash G_1(d_1)} E_{(a,c)}(|G_1(a,c)|) \\ & - \sum_{(a,c)\in G_1(d_1)\backslash G_0(d_1)} E_{(a,c)}(|G_0(a,c)|) \\ & = \sum_{(a,c)\in G_0(d_1)\backslash G_1(d_1)} E_{(a,c)}(|G_0(a,c)| - 1) \\ & - \sum_{(a,c)\in G_1(d_1)\backslash G_0(d_1)} E_{(a,c)}(|G_1(a,c)| - 1). \end{aligned}\right\} \tag{12}$$

From (9)-(12) we conclude therefore that

$$\left.\begin{aligned}
&v_{d_1}(G_1) - v_{d_1}(G_0) \\
&= \textstyle\sum_{(a,c)\in G_1(d_1)\backslash G_0(d_1)} r_{d_1}(a,c) \\
&+ \left[\textstyle\sum_{(a,c)\in G_1(d_1)\backslash G_0(d_1)} \left(I_{(a,c)}(|G_1(a,c)|) - E_{(a,c)}(|G_1(a,c)| - 1)\right)\right] \\
&- \textstyle\sum_{(a,c)\in G_0(d_1)\backslash G_1(d_1)} r_{d_1}(a,c) \\
&- \left[\textstyle\sum_{(a,c)\in G_0(d_1)\backslash G_1(d_1)} \left(I_{(a,c)}(|G_0(a,c)|) - E_{(a,c)}(|G_0(a,c)| - 1)\right)\right] \\
&= \textstyle\sum_{(a,c)\in G_1(d_1)\backslash G_0(d_1)} \left[r_{d_1}(a,c) + \Phi_{(a,c)}(|G_1(a,c)|)\right] \\
&- \left(\textstyle\sum_{(a,c)\in G_0(d_1)\backslash G_1(d_1)} \left[r_{d_1}(a,c) + \Phi_{(a,c)}(|G_0(a,c)|)\right]\right).
\end{aligned}\right\} \quad (13)$$

Next consider $P(G_1) - P(G_0)$. We have,

$$\left.\begin{aligned}
&P(G_1) - P(G_0) \\
&= \textstyle\sum_{(a,c)} \left[\sum_{d\in G_1(a,c)} r_d(a,c) + \sum_{k=0}^{|G_1(a,c)|} \Phi_{(a,c)}(k)\right] \\
&- \textstyle\sum_{(a,c)} \left[\sum_{d\in G_0(a,c)} r_d(a,c) + \sum_{k=0}^{|G_0(a,c)|} \Phi_{(a,c)}(k)\right].
\end{aligned}\right\} \quad (14)$$

For $G \in \mathbb{K}$ let

$$H(G) := \{(a,c) \in A \times C : (a,(d,c)) \in G \text{ for some } d \in D\},$$

and note that

$$H(G_1)\backslash G_1(d_1) \cup G_0(d_1) = H(G_0)\backslash G_1(d_1) \cup G_0(d_1) := S.$$

Now observe that

$$\left.\begin{aligned}
&\textstyle\sum_{(a,c)} \sum_{d\in G_1(a,c)} r_d(a,c) - \sum_{(a,c)} \sum_{d\in G_0(a,c)} r_d(a,c) \\
&= \textstyle\sum_{(a,c)\in S\cup(G_1(d_1)\cap G_0(d_1))} \left(\sum_{d\in G_1(a,c)} r_d(a,c) - \sum_{d\in G_0(a,c)} r_d(a,c)\right) \\
&+ \textstyle\sum_{(a,c)\in G_1(d_1)\backslash G_0(d_1)} \left(\sum_{d\in G_1(a,c)} r_d(a,c) - \sum_{d\in G_0(a,c)} r_d(a,c)\right) \\
&+ \textstyle\sum_{(a,c)\in G_0(d_1)\backslash G_1(d_1)} \left(\sum_{d\in G_1(a,c)} r_d(a,c) - \sum_{d\in G_0(a,c)} r_d(a,c)\right).
\end{aligned}\right\} \quad (15)$$

Moreover, note that

$$\left.\begin{array}{c}\sum_{(a,c)\in S\cup(G_1(d_1)\cap G_0(d_1))}\left(\sum_{d\in G_1(a,c)} r_d(a,c)-\sum_{d\in G_0(a,c)} r_d(a,c)\right)=0\\ \sum_{(a,c)\in G_1(d_1)\backslash G_0(d_1)}\left(\sum_{d\in G_1(a,c)} r_d(a,c)-\sum_{d\in G_0(a,c)} r_d(a,c)\right)\\ =\sum_{(a,c)\in G_1(d_1)\backslash G_0(d_1)} r_{d_1}(a,c)\\ \text{and}\\ \sum_{(a,c)\in G_0(d_1)\backslash G_1(d_1)}\left(\sum_{d\in G_1(a,c)} r_d(a,c)-\sum_{d\in G_0(a,c)} r_d(a,c)\right)\\ =-\sum_{(a,c)\in G_0(d_1)\backslash G_1(d_1)} r_{d_1}(a,c).\end{array}\right\}\quad(16)$$

Therefore,

$$\left.\begin{array}{c}\sum_{(a,c)}\sum_{d\in G_1(a,c)} r_d(a,c)-\sum_{(a,c)}\sum_{d\in G_0(a,c)} r_d(a,c)\\ =\sum_{(a,c)\in G_1(d_1)\backslash G_0(d_1)} r_{d_1}(a,c)-\sum_{(a,c)\in G_0(d_1)\backslash G_1(d_1)} r_{d_1}(a,c).\end{array}\right\}\quad(17)$$

Next, observe that

$$\left.\begin{array}{c}\sum_{(a,c)}\sum_{k=0}^{|G_1(a,c)|}\Phi_{(a,c)}(k)-\sum_{(a,c)}\sum_{k=0}^{|G_0(a,c)|}\Phi_{(a,c)}(k)\\ =\sum_{(a,c)\in S\cup(G_1(d_1)\cap G_0(d_1))}\left(\sum_{k=0}^{|G_1(a,c)|}\Phi_{(a,c)}(k)-\sum_{k=0}^{|G_0(a,c)|}\Phi_{(a,c)}(k)\right)\\ +\sum_{(a,c)\in G_1(d_1)\backslash G_0(d_1)}\Phi_{(a,c)}(|G_1(a,c)|)\\ -\sum_{(a,c)\in G_0(d_1)\backslash G_1(d_1)}\Phi_{(a,c)}(|G_0(a,c)|).\end{array}\right\}\quad(18)$$

Moreover, given (9),

$$\sum_{(a,c)\in S\cup(G_1(d_1)\cap G_0(d_1))}\left(\sum_{k=0}^{|G_1(a,c)|}\Phi_{(a,c)}(k)-\sum_{k=0}^{|G_0(a,c)|}\Phi_{(a,c)}(k)\right)=0.\quad(19)$$

From (15)-(19) we conclude that

$$\left.\begin{array}{l}P(G_1)-P(G_0)\\ =\sum_{(a,c)\in G_1(d_1)\backslash G_0(d_1)} r_{d_1}(a,c)+\sum_{(a,c)\in G_1(d_1)\backslash G_0(d_1)}\Phi_{(a,c)}(|G_1(a,c)|)\\ -\sum_{(a,c)\in G_0(d_1)\backslash G_1(d_1)} r_{d_1}(a,c)-\sum_{(a,c)\in G_0(d_1)\backslash G_1(d_1)}\Phi_{(a,c)}(|G_0(a,c)|)\\ =\sum_{(a,c)\in G_1(d_1)\backslash G_0(d_1)}\left[r_{d_1}(a,c)+\Phi_{(a,c)}(|G_1(a,c)|)\right]\\ -\left(\sum_{(a,c)\in G_0(d_1)\backslash G_1(d_1)}\left[r_{d_1}(a,c)+\Phi_{(a,c)}(|G_0(a,c)|)\right]\right).\end{array}\right\}\quad(20)$$

Therefore, for all noncooperative changes $G_0\rightarrow_{d_1} G_1$,

$$v_{d_1}(G_1)-v_{d_1}(G_0)=P(G_1)-P(G_0).$$

■

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