Existence of Nash Equilibrium in Competitive Nonlinear Pricing Games with Incomplete Information

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Abstract

We model strategic competition in a market with asymmetric information as a noncooperative game in which each seller competes for a buyer of unknown type by offering the buyer a catalog of products and prices. We call this game a *catalog game*. Our main objective is to show that catalog games have Nash equilibria. The Nash existence problem for catalog games is particularly contentious due to payoff discontinuities caused by tie-breaking. We make three contributions. First, we establish under very mild conditions on primitives that no matter what the tie-breaking rule, catalog games are *uniformly payoff secure*, and therefore have mixed extensions which are payoff secure. Second, we show that if the tie-breaking rule awards the sale to firms which value it most (i.e., breaks ties in favor of firms which stand to make the highest profit), then firm profits are reciprocally upper semicontinuous (i.e., the mixed catalog game is reciprocally upper semincontinuous). This in turn implies that the mixed catalog game satisfies Reny's condition of better-reply security - a condition sufficient for existence (Reny (1999), *Econometrica* 67, 1029-1056). Third, we show by example that if the tie-breaking rule does not award the sale to firms which value it most (for example, if ties are broken randomly with equal probability), then the catalog game has no Nash equilibrium.

KEYWORDS: competitive nonlinear pricing games, discontinuous games, existence of Nash equilibrium, competitive contracting, uniform payoff security,

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reciprocal upper semicontinuity, better-reply security. JEL Classification: C6, C7, D4

1 Introduction

Firms often compete by offering potential buyers catalogs of products and prices. Examples of catalog competition abound. Fidelity Investments Inc. competes in the mutual funds market by offering investors a catalog of funds with differing risks and fee structures (e.g., equity funds, bond funds, index funds). California competes with other states to attract businesses by offering a catalog of amenities and taxes. Intel competes with other high tech firms to attract top engineers by offering a catalog of compensation and benefit packages. Cell phone companies compete by offering a plethora of calling plans. And the list could go on. A common feature of all of these examples is the presence of asymmetric information. Firms do not know all of the relevant characteristics of potential buyers. By offering a catalog each firm is better able to screen potential buyers by allowing buyers to sort themselves. Moreover, by offering a well-chosen catalog of products and prices a firm may be able to deter a competitor from defecting to a new catalog or to prevent such a defection from eroding the firm's customer base.

In this paper we model strategic competition in a market with asymmetric information as a noncooperative game in which each firm competes for the business of a buyer of unknown type by offering the buyer a catalog of products and prices. Thus each firm's strategy space is a set of catalogs. The buyer's type parameter, known to the buyer at the time the buyer chooses a firm and a contract, is known only up to a distribution by firms at the time firms choose their catalogs.[1] The timing in our model is Stackelberg: in the first stage, given the distribution of buyer types known to all firms, firms simultaneously and noncooperatively choose their catalog offers. In the second stage the buyer, knowing his type, chooses a single firm and product-price pair from that firm's catalog. By backward induction, our Stackelberg game with asymmetric information reduces to a game over catalogs played by firms where each firm's expected payoff (with respect to the buyer's type distribution) depends on the catalogs offered by all firms in the first stage as well as on the optimizing behavior of the buyer in the second stage. We call this induced first stage game played by firms over catalogs a *catalog game*. Our main objective is to show that catalog games have Nash equilibria. In the game-theoretic model we develop each firm begins with a compact set of products and a closed, bounded interval of prices and from this feasible set of product-price pairs chooses a nonempty, closed subset of products and prices - a catalog - to offer the buyer. Our model is applicable to a wide variety of competitive situations. For example, the game can be specialized to model simultaneous price and product competition in the mutual funds market by

[1] We can also interpret the model as one in which several firms compete in a market populated by many buyers where each buyer is identified (or indexed) by a random type parameter and where firms at the time they choose their catalogs only know the distribution of the type parameter in the population of buyers.

letting the set of products be a space of functions where each function represents the state-contingent returns of a specific fund the firm might offer. Letting the space of products be a closed bounded subset of R^L, the game can be specialized to model simultaneous competition in price and multidimensional product characteristics such as quality, quantity, and location.[2]

Two complications arise in proving the existence of a Nash equilibrium. First, because the space of catalogs (also a compact metric space) is not a vector space, the usual method of proving existence via a fixed point argument is not available. Thus, in order to address the existence question, we must introduce mixed strategies (or probabilistic strategies) over catalogs and consider the mixed extension of the catalog game (i.e., the mixed catalog game). Second, catalog games, as well as their mixed extensions, are discontinuous. The payoff discontinuities problem is particularly contentious due to the problem of tie-breaking. In Page and Monteiro (2003), the Nash problem for catalog games is analyzed assuming that each firm behaves as if all ties will be broken in the firm's favor - that is, assuming that whenever the buyer is indifferent between the catalogs of two or more firms or indifferent between two or more contracts in a given catalog, the firm computes expected profit assuming that the tie will be resolved in the firm's favor. Under this optimistic tie-breaking rule each firm's expected profit function has desirable properties, including upper semicontinuity. In this paper, we consider the Nash problem under alternative assumptions concerning tie-breaking. We make three contributions to the literature on the existence of Nash equilibrium in discontinuous games. First, we establish under very mild conditions on primitives that no matter what the tie-breaking rule, catalog games are *uniformly payoff secure*,[3] and therefore by Monteiro and Page (2006) catalog games, no matter what the tie-breaking rule, have mixed extensions which are payoff secure.[4] Second, we show that if the tie-breaking rule awards the sale to firms which value it most (i.e., breaks ties in favor of firms which stand to make the highest profit), then firm profits are reciprocally upper semicontinuous (i.e., the mixed catalog game is reciprocally

[2] Thus, our catalog game covers as special cases, Bertrand price competition, Cournot quantity competition, and Hotelling competition.

[3] A catalog game is uniformly payoff secure if for every firm j, every catalog C_j^*, and every $\varepsilon > 0$, there exists a catalog $\widehat{C}_j$ such that for any catalogs C_{-j} offered by other firms there is a neighborhood of C_{-j} such that

$$\Pi_j(\widehat{C}_j, C'_{-j}) \geq \Pi_j(C_j^*, C_{-j}) - \varepsilon,$$

for all deviations C'_{-j} in that neighborhood of C_{-j}. Thus, the catalog game is uniformly payoff secure if for each firm j and each starting catalog strategy C_j^* there is a defensive catalog strategy, $\widehat{C}_j$, that firm j can move to in order to secure an expected payoff of at least

$$\Pi_j(C_j^*, C_{-j}) - \varepsilon$$

given any starting strategy profile, C_{-j}, of other firms and any deviation by other firms to strategies, C'_{-j}, in a neighborhood of C_{-j}. A formal definition of uniform payoff security is given in section 3.1.

[4] Following Reny (1999), a game is said to be payoff secure if for every joint strategy, $x = (x_i, x_{-i})$, each agent i has a strategy x_i^* that virtually guarantees the payoff he receives at x even if other agents play strategies, x'_{-i}, slightly different from x_{-i}. Uniform payoff security of course implies payoff security, but the reverse implication does not hold in general.

upper semicontinuous).[5] Reciprocal upper semicontinuity together with payoff security in turn imply that the mixed catalog game is better-reply secure - a condition sufficient for existence (see Reny (1999)).[6] Third, we show by example that if the tie-breaking rule does not award the sale to firms which value it most (for example, if ties are broken randomly with equal probability), then the catalog game has no Nash equilibrium. Thus, we show by example that not all reasonable tie-breaking rules admit Nash equilibria. The fundamental problem is that while mixed catalog games are payoff secure under any tie-breaking rule (due to the uniform payoff security of the underlying game), mixed catalog games are not necessarily reciprocally upper semicontinuous under any tie-breaking rule. Thus, our general result establishing the uniform payoff security of catalog games for any tie-breaking rule allows us to reduce the Nash problem for mixed catalog games to a search for tie-breaking rules which guarantee reciprocal upper semicontinuity. In our search, we show that in general an efficiency rule[7] (i.e., any tie-breaking rule which awards the sale to firms which value it most) guarantees reciprocal upper semicontinuity and therefore guarantees the existence of a Nash equilibrium - and we show by an example that all other rules may fail to guarantee existence.

2 A Model of Catalog Competition

We construct a model in which m firms, indexed by i *and* j $(= 1, 2, \ldots, m)$, compete for the business of a buyer of unknown type $t \in T$ via the catalogs of products and prices firms offer to the buyer. In our model, firms move first, simultaneously choosing their catalogs. The buyer moves second, choosing for each of his types a firm with which to do business and a product-price pair from that firm's catalog.

2.1 Primitives

2.1.1 Buyer Types, Sales Contracts, and Catalogs

We shall assume that

(A-1) the set of buyer types is given by a probability space, $(T, B(T), \mu)$, where T is a Borel space, $B(T)$ is the Borel σ-field in T, and μ is a probability measure defined on $B(T)$.

Recall that a Borel space is a Borel subset of a complete separable metric space. Under (A-1), multidimensional type descriptions are allowed.

[5] Informally, a game is reciprocally upper semicontinuous if, whenever some agent's payoff jumps down, some other agent's payoff jumps up (see Simon (1987), Dasgupta and Maskin (1986), and Reny (1999)). Reciprocal upper semicontinuity is implied by upper semicontinuity (but the converse does not hold in general).

[6] A game is said to be better-reply secure if for every nonequilibrium strategy, x^*, and every payoff vector limit u^*, generated by strategies approaching x^* some player has a strategy yielding a payoff strictly above u_i^* even if other players deviate slightly from x^*.

[7] The motivation for using the term "efficiency rule" comes from auction theory where an auction is said to be efficient if awards the object to bidders who value it most.

Let Y be a set representing all possible products firms can offer buyers and let D be a subset of the real numbers R representing the prices firms might charge. Each firm $i = 1, 2, \ldots, m$ is constrained to construct his catalog from a feasible set of products and prices given by

$$K_i := Y_i \times D,$$

where $Y_i \subseteq Y$ is the set of all possible products that firm i can offer to the buyer. Given feasible sets K_i, $i = 1, 2, \ldots, m$, each firm then competes by offering the buyer a nonempty closed subset C_i of products and prices from feasible set K_i. We call such a nonempty closed subset a catalog. Each firm's *feasible set of catalogs* is then given by the collection of *all nonempty, closed subsets of* K_i, denoted by $P_f(K_i)$. Elements of catalog C_i, denoted by (x, p), can be viewed as describing the relevant characteristics of the sales contracts offered by firm i. For example, if Y_i is a subset of R^L, so that catalog $C_i \in P_f(K_i)$ is a closed subset in $R^L \times R$, then in sales contract $(x, p) \in C_i$, $x = (x_1, \ldots, x_L) \in Y_i$ might describe product characteristics such as quantity, quality, and location, while $p \in D$ gives the price. Alternatively, the vector x might describe a bundle of products offered by firm i at price p.

We shall assume that,

(A-2) (i) Y is a compact metric space, containing an element 0 which we shall agree denotes "no contracting," (ii) Y_i is a nonempty closed subset of Y containing 0, and (iii) D is a closed bounded interval of the nonnegative reals containing 0.

Since $K_i := Y_i \times D$ is a compact metric space, the collection of catalogs, $P_f(K_i)$, equipped with the Hausdorff metric h is automatically compact (see Aliprantis and Border (1999) for the definition of the Hausdorff metric and a discussion).

Figure 1 depicts a catalog firm i might offer the buyer.

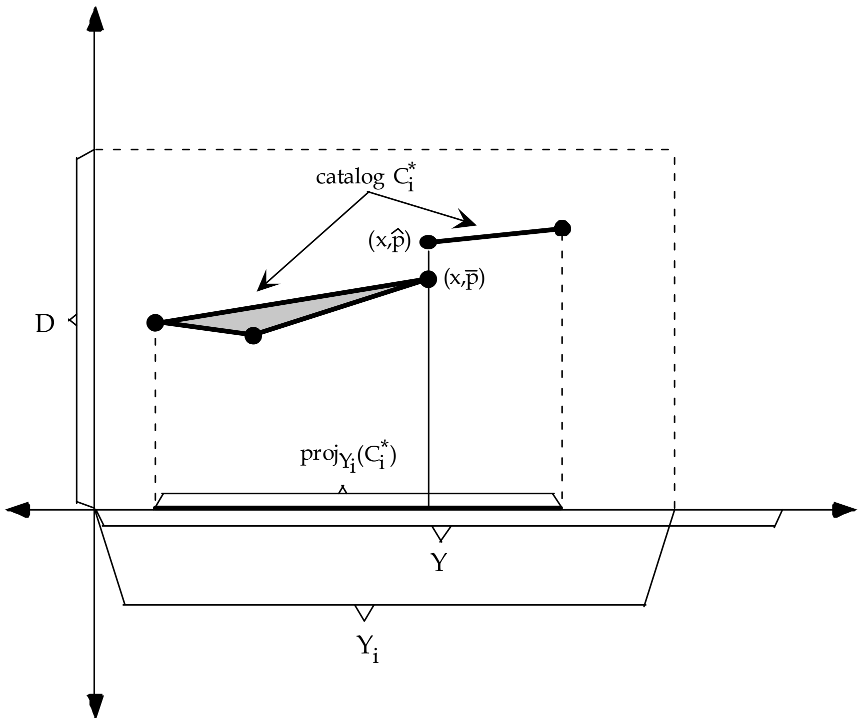

Figure 1

Note that firm $i's$ feasible product set Y_i is a proper subset of the set Y of all possible products firms can offer. Also note that under our specification of each firm's strategy space as the space of all possible catalogs, $P_f(K_i)$, we do not rule out the possibility that firms offer products at multiple prices. In catalog $C_i^* \in P_f(K_i)$ depicted in Figure 1 many products are offered by firm i at multiple prices. For example, product x is offered at prices $\widehat{p}$ and $\overline{p}$. Firm $i's$ "product line" is given by the projection, $proj_{Y_i}(C_i^*)$, of catalog C_i^* onto firm $i's$ feasible product set Y_i. Finally, note that firm $i's$ product line is a proper subset of firm $i's$ feasible product set Y_i.

Firm i can opt out of the selling game - i.e., the catalog game - by offering the "no contracting" catalog, $\{(0,0)\} \in P_f(K_i)$. Thus, firm participation is endogenous.

Firms' strategy choices can be summarized via a catalog profile,

$$(C_1, \ldots, C_m), \tag{1}$$

where the i^{th} component of the m-tuple $(C_1, \ldots, C_m)$ is the catalog offered by the i^{th} firm. We shall denote by

$$\mathbf{P} := P_f(K_1) \times \cdots \times P_f(K_m)$$

the space of all catalog profiles. If $\mathbf{P}$ is equipped with the metric $h_{\mathbf{P}}$ given by

$$h_{\mathbf{P}}\left((C_1, \ldots, C_m), (C'_1, \ldots, C'_m)\right) := \max\{h(C_i, C'_i) : i = 1, 2, \ldots, m\}, \tag{2}$$

then the space of catalog profiles $(\mathbf{P}, h_{\mathbf{P}})$ is a compact metric space.

2.1.2 The Buyer's Constraint Set

In order to allow for the possibility that the buyer may wish to abstain from contracting altogether, we assume that there is a fictitious firm $i = 0$ with feasible set of products and prices K_0 given by

$$K_0 := \{(0,0)\}\,. \tag{3}$$

Thus, for fictitious firm 0, $P_f(K_0) = \{(0,0)\}$

Letting $I = \{0, 1, 2, \ldots, m\}$, if firms $i = 1, \ldots, m$ offer catalog profile $C := (C_1, \ldots, C_m) \in \mathbf{P}$, then the buyer's constraint set is given by

$$\Gamma(C_1, \ldots, C_m) := \{(i, x, p) \in I \times Y \times D : (x, p) \in C_i\}. \tag{4}$$

If the buyer chooses firm-contract pair $(i, x, p) \in \Gamma(C_1, \ldots, C_m)$ with $(x, p) \in C_i$, $i \neq 0$, this means that the buyer has chosen sales contract (x, p) from the catalog C_i of firm i, while $(i, x, p) = (0, 0, 0)$ indicates that the buyer has chosen to abstain from contracting altogether (i.e., has chosen contract $(0, 0)$ from the fictitious firm).[8] Note that for all catalog profiles $C \in \mathbf{P}$ the buyer's choice set $\Gamma(C)$ is a closed subset of the compact set $I \times Y \times D$.[9]

2.1.3 The Buyer's Utility Function

We shall assume that

(A-3) the buyer's utility function $v(t, \cdot, \cdot, \cdot) : I \times Y \times D \to R$ is given by

$$v(t, i, x, p) = u(t, i, x) - p, \tag{5}$$

where, (i) for each buyer type $t \in T$, $u(t, \cdot, \cdot)$ is continuous in (i, x) and for each $(i, x) \in I \times Y$, $u(\cdot, i, x)$ is $B(T)$-measurable, (ii) $u(t, i, 0) \leq u(t, 0, 0)$ for all $t \in T$ and $i = 1, 2, \ldots, m$, and (iii) for each $i \in I$ the family of functions $U_i := \{u(t, i, \cdot) : t \in T\}$ is equicontinuous (i.e., for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $d(x, x') + |p - p'| < \delta$ then $|u(t, i, x') - p' - (u(t, i, x) - p)| < \varepsilon$ for all $t \in T$).

Note that we allow the buyer's utility to depend not only on the sales contract (x, p) but also on brand name i (i.e., the name of the firm with which the buyer contracts). However, by (A-3)(ii) if the buyer is to derive any utility from a firm's brand name beyond the reservation level, $v(t, 0, 0, 0)$, then the buyer must enter into

[8] The buyer is constrained only by the catalog offers of firms. Thus, as in the literature on monopoly nonlinear pricing, there is no formal budget constraint - other than the disutility of price (see, for example, Maskin and Riley (1984), McAfee and McMillan (1988), and Armstrong (1996)).

[9] Equip I with the discrete metric $d_I(\cdot, \cdot)$ given by

$$d_I(i, i') = \begin{cases} 1 & \text{if } i \neq i' \\ 0 & \text{otherwise.} \end{cases}$$

a contract with the firm. Allowing utility to depend on brand names *does not* rule out the possibility that some (or all) types of the buyer are completely indifferent to brand names. Also note that assumption (A-3)(iii), equicontinuity, will be satisfied automatically if the set of buyer types T is compact and for each $i \in I$, $u(\cdot, i, \cdot)$ is continuous on $T \times Y$.[10]

2.1.4 The Firm's Profit Function

We shall assume that

(A-4) the jth firm's profit function, $\pi_j(\cdot, \cdot, \cdot) : I \times Y \times D \to R$ is given by

$$\pi_j(i, x, p) := (p - c_j(x))\, I_j(i), \tag{6}$$

where

$$I_j(i) = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ otherwise,} \end{cases}$$

and where the cost function $c_j(\cdot) : Y \to R_+$ is nonnegative and lower semicontinuous.

Under (A-4) the j^{th} firm's profit function $\pi_j(\cdot, \cdot, \cdot)$ is upper semicontinuous on the compact set $I \times Y \times D$. Note that each firm's profit function does not depend directly on the buyer's type. Also note that if the buyer chooses not to contract with firm j, then firm j's profit is zero.

2.2 The Stackelberg Game of Catalog Competition

2.2.1 The Buyer's Problem: The Second Stage of the Stackelberg Game

Given catalog profile $C := (C_1, \ldots, C_m) \in \mathbf{P}$ chosen by firms in the first stage, a type t buyer's choice problem in the second stage is given by

$$\max \{v(t, i, x, p) : (i, x, p) \in \Gamma(C)\}. \tag{7}$$

Note that the buyer's private information as represented by the type parameter $t \in T$ determines the buyer's preferences over firms, products, and prices, $I \times Y \times D$, and thus determines the buyers choice of a firm i and a contract (x, p) from the constraint set $\Gamma(C_1, \ldots, C_m)$ determined by the catalog profile $(C_1, \ldots, C_m)$ offered by firms.

Under assumptions (A-1)-(A-3), for each t the buyer's choice problem has a solution. Let

$$v^*(t, C_1, \ldots, C_m) := \max \{v(t, i, x, p) : (i, x, p) \in \Gamma(C_1, \ldots, C_m)\} \tag{8}$$

[10] For example, (A-3)(iii) (equicontinuity) will be satisfied automatically if for each $i \in I$, $u(\cdot, i, \cdot)$ is continuous on $T \times Y$ and the set of agent types is given by

$$T = [\underline{t}_1, \overline{t}_1] \times \cdots \times [\underline{t}_q, \overline{t}_q],$$

where $\underline{t}_c$ and $\overline{t}_c$, $c = 1, \ldots, q$, are real numbers.

and

$$\Phi(t, C_1, \ldots, C_m) \\ := \{(i, x, p) \in \Gamma(C_1, \ldots, C_m) : v(t, i, x, p) = v^*(t, C_1, \ldots, C_m)\}. \tag{9}$$

The set-valued mapping

$$(C_1, \ldots, C_m) \to \Phi(t, C_1, \ldots, C_m)$$

is a type t buyer's best response mapping. For each catalog profile

$$(C_1, \ldots, C_m) \in P_f(K_1) \times \cdots \times P_f(K_m),$$

$$\Phi(t, C_1, \ldots, C_m) \text{ is a nonempty closed subset of } I \times Y \times D.$$

The following Proposition summarizes the continuity and measurability properties of the mappings, Γ and Φ, and the optimal utility function, v^*.

Proposition *(Continuity and measurability properties): Suppose assumptions (A-1)-(A-3) hold. Then the following statements are true. (a) The choice correspondence* $\Gamma(\cdot, \ldots, \cdot)$ *is* $h_{\mathbf{P}}$*-continuous on the space of catalog profiles* $\mathbf{P}$ *(i.e., is continuous with respect to the metric* $h_{\mathbf{P}}$*), (b) The function* $v^*(\cdot, \cdot, \ldots, \cdot)$ *is* $h_{\mathbf{P}}$*-continuous on* $\mathbf{P}$ *for each* $t \in T$*, and is* $B(T)$*-measurable on* T *for each* $(C_1, \ldots, C_m) \in \mathbf{P}$*. (c) For each* $t \in T$*,* $\Phi(t, \cdot, \ldots, \cdot)$ *is* $h_{\mathbf{P}}$ *-upper semicontinuous on* $\mathbf{P}$ *and* $\Phi(\cdot, \cdot, \ldots, \cdot)$ *is* $B(T) \times B(\mathbf{P})$*-measurable on* $T \times \mathbf{P}$*.*[11]

The proof of the Proposition above follows from Propositions 4.1 and 4.2 in Page (1992).

2.2.2 The Firms' Catalog Game: The First Stage of the Stackelberg Game

The buyer's best response mapping (9) induces in the first stage of the Stackelberg game a simultaneous move catalog game to be played by firms over their strategy spaces $P_f(K_i)$. In order to complete our description of the catalog game, we must specify each firm's expected payoff function, and in order to accomplish this, we must specify how ties are broken.

Our starting point is the maximal profit function. For buyer type $t \in T$ and catalog profile $(C_1, \ldots, C_m) \in \mathbf{P}$, the j^{th} firm's maximal profit is given by

$$\pi_j^*(t, C_1, \ldots, C_m) = \max \{\pi_j(i, x, p) : (i, x, p) \in \Phi(t, C_1, \ldots, C_m)\}, \tag{10}$$

where $\pi_j(\cdot, \cdot, \cdot)$ is the profit function specified in assumption (A-4). The quantity, $\pi_j^*(t, C_1, \ldots, C_m)$, is the maximum profit attainable by firm j given buyer type t and catalog profile $(C_1, \ldots, C_m)$ whenever all ties (i.e., ties between firms and ties

[11] Here $B(\mathbf{P})$ denotes the Borel σ-field in the compact metric space $(\mathbf{P}, h_{\mathbf{P}})$. Moreover,

$$B(\mathbf{P}) = B(P_f(K_1)) \times \cdots \times B(P_f(K_m)),$$

where $B(P_f(K_j))$ denotes the Borel σ-field in the compact metric space of catalogs $(P_f(K_j), h)$ (see Aliprantis and Border (1999) Theorem 4.43, p. 146).

within catalogs) are broken in favor of firm j.[12] Given assumption (A-4) and given the upper semicontinuity and measurability properties of the best response mapping (see the Proposition above), it follows from Proposition 4.3 in Page (1992) that the maximal profit function given in expression (10) is upper semicontinuous on $\mathbf{P}$ and $B(T) \times B(\mathbf{P})$-measurable on $T \times \mathbf{P}$ (see Page (1992), p. 275).

To complete our model of tie-breaking, we must specify a probabilistic rule for determining which firm involved in a tie, wins the tie. First, let

$$H(t, C_1, \ldots, C_m) := \{i \in I : \exists\ (x, p) \in Y \times D \text{ such that } (i, x, p) \in \Phi(t, C_1, \ldots, C_m)\}. \tag{11}$$

If $i \in H(t, C_1, \ldots, C_m)$, this means there is a sales contract, $(x, p) \in C_i$, offered by firm i that is optimal for a buyer of type t. Thus, the set $H(t, C_1, \ldots, C_m)$ is the set of firms over which a type t buyer is indifferent given catalog profile $(C_1, \ldots, C_m)$. Second, assume that

(A-5) the tie-breaking rule is given by a $B(T) \times B(\mathbf{P})$-measurable function

$$(t, C) \to (\rho_1(t, C), \ldots, \rho_m(t, C)) \tag{12}$$

such that for all $(t, C) \in T \times \mathbf{P}$,

$$0 \leq \rho_j(t, C) \leq 1 \text{ and } \sum_{j=1}^{m} \rho_j(t, C) = 1,$$

and

$$\rho_j(t, C) = \begin{cases} 0 & \text{if } j \notin H(t, C), \\ 1 & \text{if } H(t, C) = \{j\}. \end{cases}$$

Given any buyer type $t \in T$ and catalog profile $C = (C_1, \ldots, C_m) \in \mathbf{P}$, $\rho_j(t, C) \in [0, 1]$ is the probability that firm j wins the tie. Two examples of tie-breaking rules are the following:

The equal probability rule: The equal probability rule is given by

$$\rho_j(t, C) = \frac{I_{H(t,C)}(j)}{|H(t, C)|}, \tag{13}$$

where $I_{H(t,C)}(j) = 1$ if $j \in H(t, C)$ and 0 otherwise and where

$$|H(t, C_1, \ldots, C_m)| = \text{the number of firms contained in } H(t, C_1, \ldots, C_m).$$

Thus, under the equal probability rule all firms involved in a tie have an equal chance of winning the tie.

[12] We say that a tie is broken in favor of firm j if (i) whenever the buyer is indifferent between two or more catalogs (i.e., whenever there is a tie between firms), the buyer chooses from the catalog preferred and recommended by firm j and (ii) whenever the buyer is indifferent between two or more sales contracts contained in any particular catalog (whenever there is a tie between contracts within a catalog), the buyer chooses the sales contract preferred and recommended by firm j.

Efficiency rules: An efficiency rule is any probabilistic tie-breaking rule satisfying (A-5) such that in addition

$$\rho_j(t, C) = 0 \text{ if } j \in H(t, C) \text{ and } \pi_j^*(t, C) < \max_i \pi_i^*(t, C). \tag{14}$$

Thus, under an efficiency rule only firms that value the sale the highest (i.e., firms that will make the highest profit from the sale) have a chance of winning the tie. As we will show below, efficiency rules are critical to guaranteeing the existence of Nash equilibrium.

Given catalog profile $C = (C_1, \ldots, C_m) \in \mathbf{P}$, under tie breaking rule $\rho(\cdot, \cdot) := (\rho_1(\cdot, \cdot), \ldots, \rho_m(\cdot, \cdot))$, the j^{th} firm's expected payoff is given by

$$\Pi_j(C) = \int_T \rho_j(t, C)\pi_j^*(t, C)d\mu(t). \tag{15}$$

The j^{th} firm's expected maximal profit is given by

$$\Pi_j^*(C) = \int_T \pi_j^*(t, C)d\mu(t). \tag{16}$$

While the expected maximal profit functions, $\Pi_j^*(\cdot)$, are upper semicontinuous on $\mathbf{P}$, depending on the tie-breaking rule, the expected payoff functions, $\Pi_j(\cdot)$, specified in expression (15) may or may not even be reciprocally upper semicontinuous (e.g., the sum function $\sum_j \Pi_j(\cdot)$ may not be upper semicontinuous).[13] To see this, consider the following example with the equal probability rule.

Example 1 *(The Failure of Reciprocal u.s.c. under the equal probability rule): Suppose there are two firms and consider the sequence of catalogs given by*

$$\{C^n\}_{n=1}^{\infty} = \{(C_1^n, C_2^n)\}_{n=1}^{\infty} = \left\{ \left(1, 1 - \frac{1}{n}\right), (1, 1) \right\}_{n=1}^{\infty}.$$

Thus, for each n firm 1 offers a catalog $C_1^n = \left(1, 1 - \frac{1}{n}\right)$ *consisting of a single product-price pair given by* $(x, p) = (1, 1 - \frac{1}{n})$*, while firm 2 offers a catalog* $C_1^n = (1, 1)$ *consisting of a single product-price pair given by* $(x, p) = (1, 1)$*. Also, suppose that firm 1's cost function is given by* $c_1(x) = 0$ *for all x (i.e., firm 1 produces at zero cost), while firm 2's cost function is given by* $c_2(x) = x$*. Finally, suppose that the buyer has utility function given by*

$$v(t, i, x, p) = x - p.$$

For all n, the buyer chooses from firm 1.Under the equal probability rule,

$$(\rho_1(t, C^n), \rho_2(t, C^n)) = (1, 0) \textit{ for all } n,$$

[13] The upper semicontinuity of $\Pi_j^*(\cdot)$ on $\mathbf{P}$ follows from Fatou's Lemma (see Aliprantis and Border (1999), p. 407).

and we have

$$\Pi_1\left(C_1^n, C_2^n\right) = 1 - \tfrac{1}{n} \to 1$$
$$\textit{while}$$
$$\Pi_2\left(C_1^n, C_2^n\right) = 0 \textit{ for all } n.$$

In the limit

$$\left(C_1^n, C_2^n\right) \to \left\{\left(\bar{C}_1, \bar{C}_2\right)\right\} = \left\{(1,1), (1,1)\right\},$$

and the buyer is indifferent between firms 1 and 2. Thus, in the limit

$$(\rho_1(t,\bar{C}), \rho_2(t,\bar{C})) = (\frac{1}{2}, \frac{1}{2}),$$

and we have $\Pi_1\left(\bar{C}_1, \bar{C}_2\right) = \frac{1}{2} < 1$ *and* $\Pi_2\left(\bar{C}_1, \bar{C}_2\right) = 0$ *- and thus reciprocal u.s.c. fails because in the limit firm 1's profit jumps down from* 1 *to* $\frac{1}{2}$ *but there is no corresponding jump up in firm 2's profit.*

Given the catalog model specified in assumptions (A-1)-(A-5), the corresponding catalog game played by firms is given by

$$\left(P_f(K_j), \rho_j, \Pi_j\right)_{j=1}^m \tag{17}$$

where the set of catalogs, $P_f(K_j)$, is the j^{th} firm's strategy set, ρ_j is the j^{th} firm's part of the tie-breaking rule ρ, and Π_j is the j^{th} firm's expected payoff function, given in expression (15).

2.2.3 Mixed Catalog Games and Nash Equilibrium

For each firm $j = 1, 2, \ldots, m$, let $\Delta(P_f(K_j))$ denote the set of all probability measures defined on the feasible set of catalogs, $P_f(K_j)$. The strategy set $\Delta(P_f(K_j))$ is the j^{th} firm's mixed (or probabilistic) catalog strategy set. A mixed catalog strategy for firm j is a probability measure $\lambda_j \in \Delta(P_f(K_j))$ having the following interpretation: if firm j chooses strategy $\lambda_j \in \Delta(P_f(K_j))$, then given any Borel measurable subset of catalogs, $\mathbf{E} \in B(P_f(K_j))$, the probability that a catalog contained in $\mathbf{E}$ will be selected under strategy λ_j is $\lambda_j(\mathbf{E})$. Since the feasible set of catalogs, $P_f(K_j)$, equipped the Hausdorff metric, is a compact metric space, the mixed catalog strategy set $\Delta(P_f(K_j))$ is convex, compact, and metrizable for the topology of weak convergence of measures (see Aliprantis and Border (1999), Chapter 14).

If firms choose mixed strategy profile

$$(\lambda_1, \ldots, \lambda_m) \in \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m)),$$

then the j^{th} firm's expected payoff is given by

$$F_j\left(\lambda_1, \ldots, \lambda_m\right) := \int_{P_f(K_1)\times\cdots\times P_f(K_m)} \Pi_j\left(C_1, \ldots, C_m\right) \lambda_1(dC_1) \cdot \cdots \cdot \lambda_m(dC_m). \tag{18}$$

Following the terminology of Reny (1999), the game $\left(\Delta(P_f(K_j)), \rho_j, F_j\right)_{j=1}^m$ represents the mixed extension of the underlying catalog game $\left(P_f(K_j), \rho_j, \Pi_j\right)_{j=1}^m$. We shall refer to the game $\left(\Delta(P_f(K_j)), \rho_j, F_j\right)_{j=1}^m$ as the mixed catalog game. Note that for each firm j the expected payoff function,

$$\lambda_j \to F_j\left(\lambda_j, \lambda_{-j}\right),$$

is linear on the strategy space $\Delta(P_f(K_j))$ for each $\lambda_{-j} \in \Delta_{-j}(P_f(K_{-j}))$ (see Aliprantis and Border (1999), Theorem 14.5, p. 479).[14]

Definition 1 *(Nash Equilibrium for Mixed Catalog Games)*
A strategy profile

$$(\lambda_1^*, \ldots, \lambda_m^*) \in \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m))$$

is a Nash equilibrium for the mixed catalog game $\left(\Delta(P_f(K_j)), \rho_j, F_j\right)_{j=1}^m$ *if for all* $j = 1, 2, \ldots, m$

$$F_j\left(\lambda_j^*, \lambda_{-j}^*\right) \geq F_j\left(\lambda_j, \lambda_{-j}^*\right) \text{ for all } \lambda_j \in \Delta(P_f(K_j)).$$

If under the Nash equilibrium profile of mixed strategies $(\lambda_1^*, \ldots, \lambda_m^*)$ catalog C_j^* is selected for firm j, then catalog C_j^* is such that

$$F_j\left(C_j^*, \lambda_{-j}^*\right) = \max_{C_j \in P_f(K_j)} F_j\left(C_j, \lambda_{-j}^*\right) = \max_{\lambda_j \in \Delta(P_f(K_j))} F_j\left(\lambda_j, \lambda_{-j}^*\right) = F_j\left(\lambda_j^*, \lambda_{-j}^*\right).$$

Thus, catalog C_j^* is optimal (in an expected sense) against all catalogs which might emerge under the mixed catalog strategies, λ_{-j}^*, chosen by firm j's competitors.

3 Existence of Nash Equilibrium

Our main objective is to show that mixed catalog games have Nash equilibria. We will accomplish our objective in three steps: (1) we will show that no matter what the tie-breaking rule catalogs games are uniformly payoff secure, (2) applying Theorem 1 of Monteiro and Page (2006) we will conclude that no matter what the tie-breaking rule mixed catalog games are payoff secure, and (3) we will show that under an efficiency rule, mixed catalog games are reciprocally upper semicontinuous and therefore, by Proposition 3.2 and Corollary 5.2 in Reny (1999) have Nash equilibria. We will then show by an example that if the tie-breaking rule is not an efficiency rule, then the mixed catalog game has no Nash equilibrium.

3.1 All Catalog Games Are Uniformly Payoff Secure

The crucial first step in accomplishing our objective is to show that for all tie-breaking rules satisfying (A-5) catalog games are uniformly payoff secure. We begin with a definition followed by our fundamental result on catalog games and uniform payoff security.

[14] Here, $\Delta_{-j}(P_f(K_{-j})) = \prod_{i \neq j} \Delta(P_f(K_i))$.

Definition 2 *(Uniform Payoff Security - Pure Strategies)*

We say that the catalog game, $\left(P_f(K_j), \rho_j, \Pi_j\right)_{j=1}^m$, *is uniformly payoff secure if for every firm* $j \in \{1, \dots, m\}$, *every catalog* $C_j^* \in P_f(K_j)$, *and every* $\varepsilon > 0$, *there exists a catalog* $\widehat{C}_j \in P_f(K_j)$ *such that for every* C_{-j} *there is a neighborhood* $\mathcal{N}(C_{-j})$ *of* C_{-j} *such that for all* $C'_{-j} \in \mathcal{N}(C_{-j})$, $\Pi_j(\widehat{C}_j, C'_{-j}) \geq \Pi_j(C_j^*, C_{-j}) - \varepsilon$.

Thus, the catalog game, $\left(P_f(K_j), \rho_j, \Pi_j\right)_{j=1}^m$, is uniformly payoff secure if each firm j starting at any catalog C_j^* has a catalog $\widehat{C}_j$ the firm can move to in order to secure a payoff of $\Pi_j(C_j^*, C_{-j}) - \varepsilon$ against deviations by other firms in some neighborhood of C_{-j} for all catalog profiles $C_{-j} \in P_f(K_{-j})$ (i.e., for each firm j and each catalog C_j^* there is a catalog $\widehat{C}_j$ that provides security for all C_{-j}).

Theorem 1 *(All catalog games are uniformly payoff secure)*

Under assumptions (A-1)-(A-5), the catalog game, $\left(P_f(K_j), \rho_j, \Pi_j\right)_{j=1}^m$, *is uniformly payoff secure.*

The proof of Theorem 1 is given in section 4.

3.2 All Mixed Catalog Games Are Payoff Secure

Definition 3 *(Payoff Security - Mixed Strategies)*

We say that the mixed catalog game, $\left(\Delta(P_f(K_j)), \rho_j, F_j\right)_{j=1}^m$, *is payoff secure if for every mixed strategy profile*

$$\left(\lambda'_1, \dots, \lambda'_m\right) \in \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m)),$$

and every $\varepsilon > 0$ *there exists a* $\delta > 0$ *and mixed strategies* $\lambda_j^* \in \Delta(P_f(K_j))$, $j = 1, \dots, m$, *such that* $F_j\left(\lambda_j^*, \lambda_{-j}\right) \geq F_j\left(\lambda'_j, \lambda'_{-j}\right) - \varepsilon$ *for all* $(m-1)$*-tuples of mixed strategies* $\lambda_{-j} \in B_\delta(\lambda'_{-j})$.

Here, $B_\delta(\lambda'_{-j})$ is an open ball of radius δ centered at λ'_{-j} in the metric space of probability measures,

$$\Delta_{-j}(P_f(K_{-j})) := \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_{j-1})) \times \Delta(P_f(K_{j+1})) \times \cdots \times \Delta(P_f(K_m)).$$

It follows from Theorem 1 of Monteiro and Page (2006) that if a catalog game is uniformly payoff secure, then its mixed extension is payoff secure. This is important because while it is easy to check for uniform payoff security in the catalog game, the same cannot be said about checking for payoff security in the game's mixed extension. We have the following Theorem:

Theorem 2 *(Mixed Catalog Games Are Payoff Secure)*

Under assumptions (A-1)-(A-5), the mixed catalog game, $\left(\Delta(P_f(K_j)), \rho_j, F_j\right)_{j=1}^m$, *is payoff secure.*

Proof. A catalog game satisfying assumptions (A-1)-(A-5) is a compact metric game. The proposition follows from the more general proposition below.

Proposition 1 (Monteiro and Page (2006)) *If a compact, Hausdorff game G is uniformly payoff secure then its mixed extension $\overline{G}$ is payoff secure.*

For a proof see Monteiro and Page(2006). ■

3.3 Mixed Catalog Games and Nash Equilibrium

In his paper on Nash equilibrium in discontinuous games, Reny (1999) showed that any game with compact, convex strategy spaces and payoffs at least quasiconcave in each player's strategies has a Nash equilibrium if, in addition, the game is payoff secure and at least reciprocally upper semicontinuous. Mixed catalog games have compact, convex strategy spaces and payoffs linear in each player's strategies - and we have shown that they are payoff secure. Whether or not they are reciprocally upper semicontinuous depends on the tie-breaking rule. We begin this section by showing that under any efficiency rule, mixed catalog games are indeed reciprocally upper semicontinuous and therefore have Nash equilibria. More importantly, we present an example of a mixed catalog game in which the tie-breaking rule is *not* an efficiency rule and we show that as a consequence the game has no Nash equilibrium. We begin with a definition of reciprocal upper semicontinuity for mixed catalog games.

Definition 4 *(Reciprocal Upper Semicontinuity of Mixed Catalog Games)*
We say that the mixed catalog game, $\left(\Delta(P_f(K_j)), \rho_j, F_j\right)_{j=1}^{m}$, is reciprocally upper semicontinuous if whenever

$$(\lambda', F') = \left(\lambda'_1, \ldots, \lambda'_m, F'_1, \ldots, F'_m\right) \in (\Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m))) \times R^m$$

is contained in the closure of the graph of the vector payoff function $\lambda \to F(\lambda)$ and $F_i(\lambda') \leq F'_i$ for every firm i, then $F_i(\lambda') = F'_i$ for every firm i.

If the sum of profits, $\sum_i \Pi_i(\cdot)$, is upper semicontinuous in catalog profiles, then the mixed catalog game $\left(\Delta(P_f(K_j)), \rho_j, F_j\right)_{j=1}^{m}$ is reciprocally upper semicontinuous (see Proposition 5.1 in Reny (1999)).

We now have our main result on the existence of Nash equilibrium for mixed catalog games.

Theorem 3 *(All mixed catalog games with an efficiency rule have a Nash equilibrium)*
Under assumptions (A-1)-(A-5), the mixed catalog game, $\left(\Delta(P_f(K_j)), \rho_j, F_j\right)_{j=1}^{m}$, has a Nash equilibrium provided the tie-breaking rule

$$(t, C) \to (\rho_1(t, C), \ldots, \rho_m(t, C))$$

is an efficiency rule.

Proof. By Proposition 3.2 and Corollary 5.2 in Reny (1999), it suffices to show that under an efficiency rule the mixed catalog game is reciprocally upper semicontinuous. We will accomplish this by showing that the sum function, $\sum_i \Pi_i(\cdot)$, is upper semicontinuous. We have,

$$\sum_{j=1}^m \Pi_j(C) = \sum_{j=1}^m \int_T \rho_j(t,C)\pi_j^*(t,C)d\mu(t)$$

$$= \int_T \sum_{j=1}^m \rho_j(t,C)\pi_j^*(t,C)d\mu(t)$$

$$= \int_T \left(\sum_{j=1}^m \rho_j(t,C)\right) \max_j \pi_j^*(t,C)d\mu(t)$$

$$= \int_T \max_j \pi_j^*(t,C)d\mu(t).$$

Since $\pi_j^*(t,C)$ is upper semicontinuous, $\max_j \pi_j^*(t,C)$ is upper semicontinuous as well. Thus the game is reciprocally upper semicontinuous. ■

3.4 Tie-Breaking Rules and Nonexistence of Nash Equilibrium

We have shown that the class of mixed catalog games with tie-breaking determined by an efficiency rule has the Nash existence property (i.e., every mixed catalog game with an efficiency rule has a Nash equilibrium). However, as the following example shows, the same cannot be concluded about the complementary class.

Example 2 *(A mixed catalog game with no Nash equilibrium where tie-breaking is determined by a rule that is not an efficiency rule)*

Consider two firms competing for the business of an agent having one type and suppose that profit and utility functions are given by

$$\pi_1(i,x,p) := (p - c_j(x))\, I_j(i) = \left(p - \tfrac{x}{4}\right) I_1(i),$$

$$\pi_2(i,x,p) := (p - c_j(x))\, I_j(i) = \left(p - \tfrac{x}{2}\right) I_2(i),$$

$$v(i,x,p) = x - p.$$

Suppose also that if there is a tie, then firm 1 wins with probability $\rho \in [0,1)$ and firm 2 wins with probability $(1-\rho)$. Each firm has feasible product-price set given by

$$K_i = [0,\bar{x}] \times [0,\overline{p}] \text{ where } 0 < \bar{x} \leq \overline{p}$$

Note that a firm that charges a price $p > \bar{x}$ will not sell anything since $v(i,x,p) \leq \bar{x} - p < 0$.

We show that this economy has no mixed strategy Nash equilibrium. To obtain a contradiction, suppose that there is a mixed strategy Nash equilibrium, (λ_1,λ_2). Let

$$M^1 := \sup_{C \in P_f(K_1)} \int_{P_f(K_2)} \rho_1(C,D)\,\pi_1^*(C,D)\,d\lambda_2(D).$$

Let

$$\mathbf{C} := \left\{ C \in P_f(K_1) : \int_{P_f(K_2)} \rho_1 (C, D)\, \pi_1^* (C, D)\, d\lambda_2 (D) = M^1 \right\},$$

and note that $\lambda_1 (P_f(K_1) \backslash \mathbf{C}) = 0$*. We need a couple of definitions. Given a catalog* $C \in P_f(K_1)$*, define*

$$v(C) = \max \{x - p : (x, p) \in C\}.$$

Thus $v^* (C, D) = \max \{v(C), v(D), 0\}$*. Now define*

$$\pi_1^* (C) = \max \left\{ p - \tfrac{x}{4} : (x, p) \in C, x - p = v(C) \right\},$$

$$\pi_2^* (C) = \max \left\{ p - \tfrac{x}{2} : (x, p) \in C, x - p = v(C) \right\},$$

and let distribution functions G *and* F *be defined by*

$$G(v) = \lambda_2 \{D; v(D) \leq v\},$$

$$F(v) = \lambda_1 \{C; v(C) \leq v\}.$$

Thus $G(\bar{x}) = 1 = F(\bar{x})$ *and* $G(0-) = 0 = F(0-)$*.*[15] G *is the distribution of utilities implied by mixed strategy* λ_2 *and* F *is the distribution of utilities implied by mixed strategy* λ_1*. For every catalog* C *we have that* $\pi_1^* (C, D) = \pi_1^* (C)$ *if* $v(C) \geq \max \{v(D), 0\}$*.*[16] *Therefore,*

$$\begin{aligned}
&\textstyle\int_{P_f(K_2)} \rho_1 (C, D)\, \pi_1^* (C, D)\, d\lambda_2 (D) \\
&= \textstyle\int_{v(C) > \max\{v(D), 0\}} \pi_1^* (C)\, d\lambda_2 (D) + \int_{v(C) = \max\{v(D), 0\}} \rho \pi_1^* (C)\, d\lambda_2 (D) \\
&= \pi_1^* (C) \left(\textstyle\int_{v(C) > \max\{v(D), 0\}} d\lambda_2 (D) + \rho \int_{v(C) = \max\{v(D), 0\}} d\lambda_2 (D) \right) \\
&= \pi_1^* (C) \left(G(v(C) -) + \rho \left(G(v(C)) - G(v(C) -) \right) \right) \\
&= \pi_1^* (C) \left(\rho G(v(C)) + (1 - \rho) G(v(C) -) \right).
\end{aligned}$$

Therefore for every catalog $C' \in P_f(K_1)$ *we have*

$$\begin{aligned}
&\pi_1^* (C') \left(\rho G(v(C')) + (1 - \rho) G(v(C') -) \right) \leq M^1 \\
&= \textstyle\int \pi_1^* (C) \left(\rho G(v(C)) + (1 - \rho) G(v(C) -) \right) d\lambda_1 (C).
\end{aligned} \tag{19}$$

Define $\pi_1^* (v) = \max \left\{ p - \frac{x}{4} : x - p = v, x \leq \bar{x} \right\}$ *as the maximum profit of firm 1 if the consumer has utility level* v*. Thus* $\pi_1^* (v) = \frac{3\bar{x}}{4} - v$*. Analogously let* $\pi_2^* (v) = \frac{\bar{x}}{2} - v$ *be the maximum profit of firm 2 if the consumer has utility level* v*. Thus,*

$$\begin{aligned}
H^1 &:= \textstyle\sup_{C \in P_f(K_1)} \pi_1^* (C) \left(\rho G(v(C)) + (1 - \rho) G(v(C) -) \right) \\
&= \textstyle\max_{0 \leq v \leq \bar{x}} \left(\frac{3\bar{x}}{4} - v \right) \left(\rho G(v) + (1 - \rho) G(v-) \right).
\end{aligned}$$

[15] Here, $G(0-) = \lim_{v \uparrow 0} G(v)$ and $F(0-) = \lim_{v \uparrow 0} F(v)$.

[16] In a Nash equilibrium, if $C \in \mathbf{C}$ then necessarily $v(C) \geq 0$.

And from (19), $H^1 = M^1$*. Let*

$$V = \left\{ v : \left(\frac{3\bar{x}}{4} - v \right) \left(\rho G\left(v\right) + \left(1-\rho\right) G\left(v-\right) \right) = M^1 \right\}.$$

If $\underline{v} = \inf\left(V \cap support(G)\right)$ *we see that* $\left(\frac{3\bar{x}}{4} - \underline{v}\right) \rho G\left(\underline{v}\right) = M^1$ *and therefore* $G\left(\underline{v}\right) > 0$*. However, if* $v \downarrow \underline{v}$*, then*

$$\left(\frac{3\bar{x}}{4} - v \right) \left(\rho G\left(v\right) + \left(1-\rho\right) G\left(v-\right) \right) \to \left(\frac{3\bar{x}}{4} - \underline{v} \right) G\left(\underline{v}\right) = M^1 = \left(\frac{3\bar{x}}{4} - \underline{v} \right) \rho G\left(\underline{v}\right),$$

a contradiction since $\rho < 1$ *(i.e., since the tie-breaking rule does not award the sale to firm 1 - the firm that values it most - with probability 1).*

We see from the reasoning above that if $\rho = 1$ (i.e., if the tie-breaking rule awards the sale to the firm that values it most), then we have an equilibrium. In particular, we have a pure strategy Nash equilibrium given by $C = D = \left\{\left(\bar{x}, \frac{\bar{x}}{2}\right)\right\}$.

3.5 Catalogs and Nonlinear Pricing Schedules

Given feasible set $K_i := Y_i \times D$ a nonlinear pricing schedule for firm i is a pair $(X_i, p_i(\cdot))$ where $X_i \subseteq Y_i$ is the firms product line and $p_i(\cdot) : X_i \to D$ is the nonlinear pricing function. Catalogs and nonlinear pricing schedules are closely related. This relationship is formalized in the *competitive taxation principle* (see Page and Monteiro (2003), Theorem 3).[17] The competitive taxation principle states that given any profile of catalogs there exists a *unique* profile of nonlinear pricing schedules which generates the same product-price selections by the buyer for all of his types as the given profile of catalogs; and *conversely*, given any profile of nonlinear pricing schedules there exists a *unique* profile of catalogs which generates the same product-price selections by the buyer for all of his types as the given profile of nonlinear pricing schedules.

The basic intuition behind the competitive taxation principle can be explained via Figure 1 and Figures 2 and 3 below. If firm i offers catalog C_i^* (Figure 1), then because utility is strictly decreasing in price, if the buyer chooses from firm i then no matter what his type, the buyer will choose a product-price combination on the lower boundary of the catalog. This lower boundary, depicted in Figure 2, is the graph of the unique nonlinear pricing schedule, $(X_i^*, p_i^*(\cdot))$, generated by catalog C_i^*, where $X_i^* = proj_{Y_i}(C_i^*)$ is firm $i's$ product line and $p_i^*(\cdot) : X_i^* \to D$ is firm $i's$ pricing

[17] The name *taxation principle* originated in the work of Guesnerie on the implementation of nonlinear tax schedules (e.g., see Guesnerie (1995)). The name has continued to be used in many areas of mechanism design including competitive nonlinear pricing (also, see Rochet (1985)).

function.

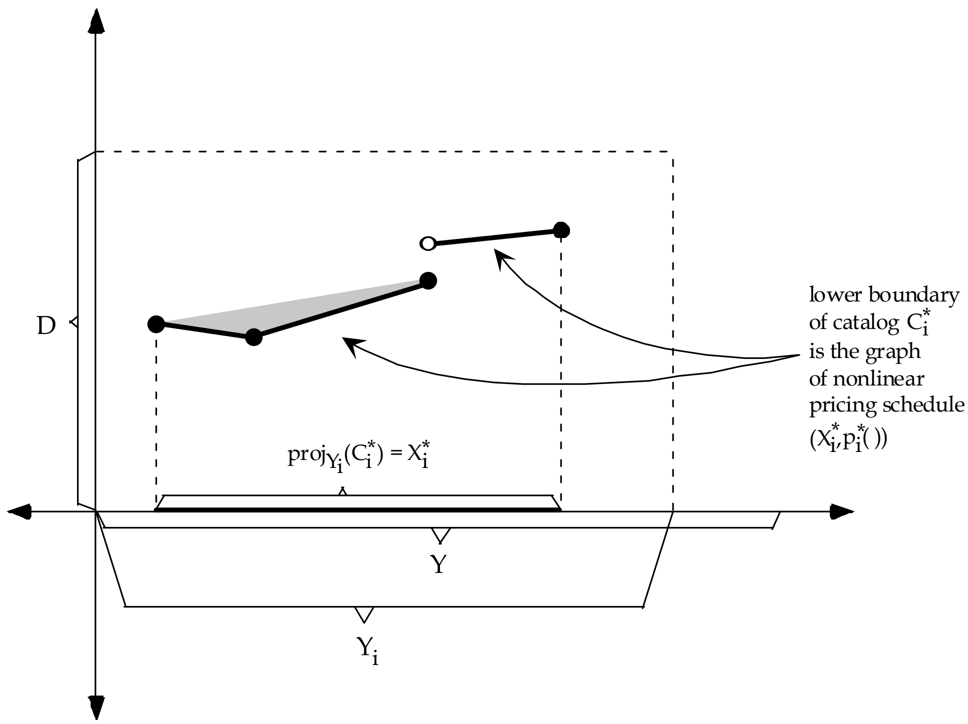

Figure 2

The nonlinear pricing function $p_i^*(\cdot)$ is given by

$$p_i^*(x) = \min\{p \in D : (x, p) \in C_i^*\},$$

for all products x contained in firm $i's$ product line X_i^*. Thus, by choosing a catalog the firm at the same time chooses a nonlinear pricing schedule - and thus, by defining mixed strategies over catalogs we are at the same time defining mixed strategies over nonlinear pricing schedules.[18]

Conversely, the closure of the graph of the pricing function $p_i^*(\cdot)$, depicted in

[18] Note that catalog C_i^* simultaneously and uniquely specifies the pricing function $p_i^*(\cdot)$ and the product line X_i^* (i.e., the domain of the nonlinear pricing function).

Figure 3, uniquely determines a catalog $\overline{C}_i^*$.[19]

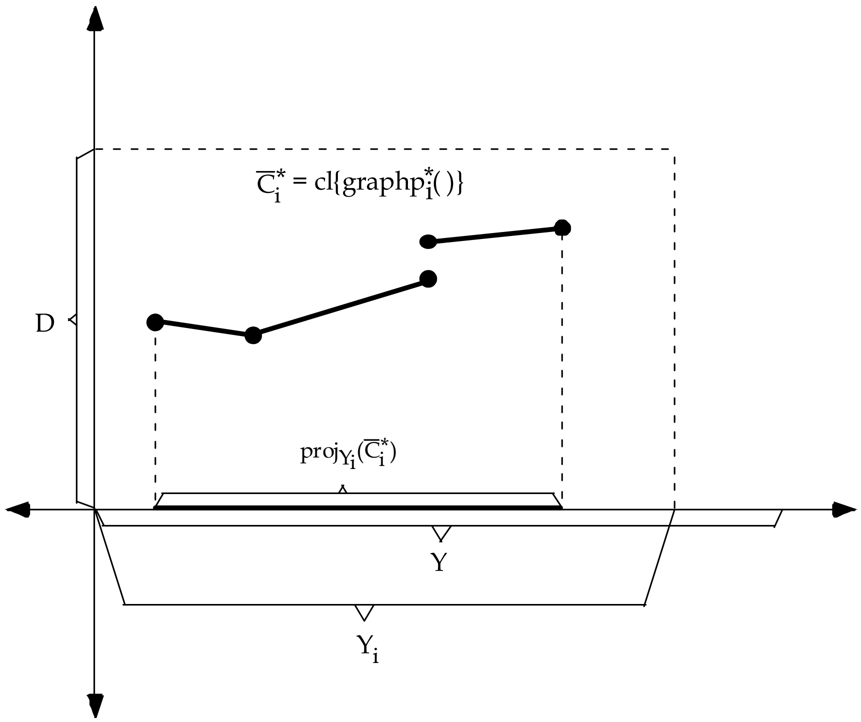

Figure 3

Note that catalogs C_i^* (in Figure 1) and $\overline{C}_i^*$ (in Figure 3) generate the same nonlinear pricing schedule, $(X_i^*, p_i^*(\cdot))$. More importantly, note that if the buyer chooses from firm i then the buyer's selections, for all his possible types, will be the same under catalogs C_i^* and $\overline{C}_i^*$, and will be equal to the buyer's selections under the nonlinear pricing schedule $(X_i^*, p_i^*(\cdot))$. Thus, the essential part of any catalog is its lower boundary - or equivalently, the essential part of any catalog is the unique nonlinear pricing schedule it generates.

If under the Nash equilibrium profile of mixed strategies, $(\lambda_1^*, \ldots, \lambda_m^*)$, catalog C_i^* is selected for firm i, then we know that catalog C_i^*, contained in the support of probability measure λ_i^*, is such that

$$F_i\left(C_i^*, \lambda_{-i}^*\right) = \max_{C_i \in P_i(K_i)} F_i\left(C_i, \lambda_{-i}^*\right) = \max_{\lambda_i \in \Delta(P_i(K_i))} F_i\left(\lambda_i, \lambda_{-i}^*\right) = F_i\left(\lambda_i^*, \lambda_{-i}^*\right).$$

Thus, the nonlinear pricing schedule generated by C_i^* is optimal (in an expected sense) against all nonlinear pricing schedules which might emerge under the mixed catalog strategies, λ_{-i}^*, chosen by firm i's competitors, and therefore, finding a Nash equilibrium in mixed strategies over catalogs is equivalent to finding a Nash equilibrium in mixed strategies over nonlinear pricing schedules.

[19] In this case, because the pricing function $p_i^*(\cdot)$ is lower semicontinuous on X_i^* with a discontinuity at x, the closure of the graph of $p_i^*(\cdot)$ differs from the graph of $p_i^*(\cdot)$. In particular, while $(x, \widehat{p})$ is contained in the closure of the graph of $p_i^*(\cdot)$, it is not contained in the graph of $p_i^*(\cdot)$ (i.e., stated loosely, catalogs and nonlinear pricing schedules are not the same).

4 Proof of Theorem 1

The proof of Theorem 1 rests on the following Lemma.

Lemma 1 *(The UPS Lemma for Catalog Games)*

Given any catalog $C_j \in P_f(K_j)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for all C_{-j} and C'_{-j} with $h_{\mathbf{P}_{-j}}(C_{-j}, C'_{-j}) < \delta$ (i.e., such that for all catalogs C_i and C'_i belonging to firms $i \neq j$ with $h(C_i, C'_i) < \delta$),

$$\rho_j(t, C_j^{\varepsilon}, C'_{-j})\pi_j^*(t, C_j^{\varepsilon}, C'_{-j}) \geq \pi_j^*(t, C_j, C_{-j})^+ - \varepsilon \text{ for all } t \in T,$$

where $\pi_j^*(t, C_j, C_{-j})^+ := \max\left\{\pi_j^*(t, C_j, C_{-j}), 0\right\}$ *and*

$$C_j^{\varepsilon} = \{(x, p - \varepsilon) : (x, p) \in C_j \text{ and } p - \varepsilon \geq c_j(x)\} \cup \{(0, 0)\}.$$

Proof. Let $\varepsilon > 0$ be given. By assumption (A-3)(iii), there is a $\delta > 0$ such that if $d(x, x') + |p - p'| < \delta$ then for all i

$$\left|u(t, i, x') - p' - (u(t, i, x) - p)\right| < \varepsilon \text{ for all } t \in T.$$

First, suppose $\pi_j^*(t, C_j, C_{-j}) > \varepsilon$. Let $(j, x, p) \in \Phi(t, C_j, C_{-j})$ be such that

$$p - c_j(x) = \pi_j^*(t, C_j, C_{-j}) > \varepsilon.$$

Then $(x, p - \varepsilon) \in C_j^{\varepsilon}$ and

$$\pi_j^*(t, C_j^{\varepsilon}, C'_{-j}) \geq p - c_j(x) - \varepsilon = \pi_j^*(t, C_j, C_{-j}) - \varepsilon. \tag{20}$$

For $i \neq j$, let $(x', p') \in C'_i$ where catalog deviation C'_i is such that $h(C_i, C'_i) < \delta$. By the properties of the Hausdorff distance, there exists $(y, q) \in C_i$ such that $d(x', y) + |p' - q| < \delta$. Thus, we have

$$u(t, j, x) - (p - \varepsilon) \geq u(t, i, y) - (q - \varepsilon) > u(t, i, x') - p'.$$

Therefore, $(i, x', p') \notin \Phi(t, C_j^{\varepsilon}, C'_{-j})$. This means that for agent types who choose firm j given catalog profile (C_j, C_{-j}) continue to do so under catalog profile $(C_j^{\varepsilon}, C'_{-j})$ for catalog deviations C'_i, $i \neq j$, such that $h(C_i, C'_i) < \delta$ - and moreover, such agent types strictly prefer firm j. Thus, $\rho_j(t, C_j^{\varepsilon}, C'_{-j}) = 1$, and from equation (20) we have for the case in which $\pi_j^*(t, C_j, C_{-j}) > \varepsilon$

$$\rho_j(t, C_j^{\varepsilon}, C'_{-j})\pi_j^*(t, C_j^{\varepsilon}, C'_{-j}) \geq \pi_j^*(t, C_j, C_{-j})^+ - \varepsilon. \tag{21}$$

Suppose now that $\pi_j^*(t, C_j, C_{-j}) \leq \varepsilon$. Then we have

$$\rho_j(t, C_j^{\varepsilon}, C'_{-j})\pi_j^*(t, C_j^{\varepsilon}, C'_{-j}) \geq 0 \geq \pi_j^*(t, C_j, C_{-j})^+ - \varepsilon,$$

and we conclude that

$$\rho_j(t, C_j^\varepsilon, C'_{-j})\pi_j^*(t, C_j^\varepsilon, C'_{-j}) \geq \pi_j^*(t, C_j, C_{-j})^+ - \varepsilon \text{ for all } t \in T.$$

■

In light of the UPS Lemma, the proof of Theorem 1 is straightforward: Let C_j be firm $j's$ catalog and let $\varepsilon > 0$ be given. As before, let

$$C_j^\varepsilon = \{(x, p - \varepsilon) : (x, p) \in C_j, p - \varepsilon \geq c_j(x)\}.$$

Let $\delta > 0$ be such that such that if $d(x, x') + |p - p'| < \delta$ then for all i and t

$$\left| u(t, i, x') - p' - (u(t, i, x) - p) \right| < \varepsilon \ .$$

We have to compare,

$$\Pi_j(C_j^\varepsilon, C'_{-j}) = \int_T \rho_j(t, C_j^\varepsilon, C'_{-j})\pi_j^*(t, C_j^\varepsilon, C'_{-j})d\mu(t)$$

with

$$\Pi_j(C_j, C_{-j}) = \int_T \rho_j(t, C_j, C_{-j})\pi_j^*(t, C_j, C_{-j})d\mu(t)$$

Applying the UPS lemma we have for all catalog deviations C'_{-j} such that

$$h_{\mathbf{P}_{-j}}(C_{-j}, C'_{-j}) < \delta,$$

$$\Pi_j(C_j^\varepsilon, C'_{-j}) \geq \textstyle\int_T \rho_j(t, C_j, C_{-j})\pi_j^*(t, C_j, C_{-j})^+ d\mu(t) - \varepsilon$$

$$\geq \textstyle\int_T \rho_j(t, C_j, C_{-j})\pi_j^*(t, C_j, C_{-j}) d\mu(t) - \varepsilon$$

$$= \Pi_j(C_j, C_{-j}) - \varepsilon.$$

■

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