The Geometry of Arbitrage and the Existence of Competitive Equilibrium*

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Abstract

We present the basic geometry of arbitrage, and use this basic geometry to shed new light on the relationships between various no-arbitrage conditions found in the literature. For example, under very mild conditions, we show that the no-arbitrage conditions of Hart (1974) and Werner (1987) are equivalent and imply the compactness of the set of utility possibilities. Moreover, we show that

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if agents' sets of useless net trades are linearly independent, then the Hart-Werner conditions are equivalent to the stronger condition of no-unbounded-arbitrage due to Page (1987) - and, in turn, all are equivalent to compactness of the set of rational allocations. We also consider the problem of existence of equilibrium. We show, for example, that under a uniformity condition on preferences weaker than Werner's uniformity condition, the Hart-Werner no-arbitrage conditions are sufficient for existence. With an additional condition of weak no half lines - a condition weaker than Werner's no-half-lines condition - we show that the Hart-Werner conditions are both necessary *and* sufficient for existence.

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1

2 Introduction

While there is no universally agreed upon definition of arbitrage, a good definition, especially within the context of a finite dimensional exchange economy is the following:

> An arbitrage opportunity is a mutually compatible set of net trades which are utility nondecreasing and, at most, costless to make.

When unbounded short sales are allowed, as is natural in asset market models, agents' choice sets are unbounded from below, and as a consequence, unbounded and mutually compatible arbitrage opportunities can arise. In such cases, prices at which all arbitrage opportunities can be exhausted may fail to exist, and thus, equilibrium may fail to exist. Since the seminal contributions of Grandmont ((1970), (1972), (1977)), Green (1973), Hart (1974), and Werner (1987), much of the research on asset market models has focused upon conditions limiting arbitrage (i.e., no-arbitrage conditions) and upon the relationship between such conditions and the existence of equilibrium.[1]

No-arbitrage conditions found in the literature generally fall into three broad categories:

(i) *conditions on net trades*, for example, Hart (1974), Page (1987), Nielsen (1989), Page, Wooders, and Monteiro (2000), and Allouch (2002);
(ii) *conditions on prices*, for example, Green (1973), Grandmont (1977,1982), Hammond (1983), and Werner (1987).
(iii) *conditions on the set of utility possibilities (namely, compactness)*, for example, Brown and Werner (1995), Dana, Le Van, and Magnien (1999).

In all cases, the role played by conditions limiting arbitrage in general equilibrium models with short sales is to bound the economy endogenously. For example, Page and Wooders (1996) show that the condition of no-unbounded-arbitrage, introduced in Page (1987), is equivalent to compactness of the set of rational allocations, and therefore implies compactness of the set of rational utility possibilities.[2] Under additional conditions on

[1] See also, for example, Milne (1976, 1980), Hammond (1983), Page (1987), Nielsen (1989), Page and Wooders (1996), Kim (1998), Dana, Le Van, Magnien (1999), Allouch (2002), and Page, Wooders, and Monteiro (2000).

[2] Because the no-arbitrage condition of Hammond (1983) - overlapping expectations - is stated in terms of properties of the subjective probability distributions of asset returns, it is difficult to make comparisons in an abstract general equilibrium setting between Hammond's condition and other no-arbitrage conditions. Page (1987) shows that under very mild conditions on utility functions and asset return distributions, Hammond's condition of overlapping expectations is equivalent to no-unbounded-arbitrage.

the model (i.e., no-half-lines and uniformity, conditions explained in the paper), Page and Wooders (1996) and Dana, Le Van, and Magnien (1999) show that no-unbounded-arbitrage is equivalent to compactness of the set of rational utility possibilities. Allouch (1999) shows that the condition of inconsequential arbitrage (see Page, Wooders, and Monteiro (2000)) implies directly compactness of the set of rational utility possibilities.

The purpose of this paper is to expose the underlying geometric structure common to all no-arbitrage conditions, and in so doing, to shed new light on how the no-arbitrage conditions found in the literature are related and how they work to guarantee boundedness and the existence of equilibrium. Under very mild conditions on our model, we show that all the well-known no-arbitrage conditions in the literature imply compactness of the set of utility possibilities. However, some conditions, for example the conditions of Hammond (1983) (overlapping expectations) and Page (1987) (no-unbounded-arbitrage), imply compactness of utility possibilities by first implying the compactness of the set of rational allocations. Alternatively, the weaker conditions of Hart (1974) (weak-no-market-arbitrage) and Werner (1987) (no-arbitrage price system) imply directly compactness of the set of utility possibilities but allow the set of rational allocations to be unbounded. Moreover, we identify precisely the conditions on the model under which all no-arbitrage conditions are equivalent

Our starting point is a basic geometric lemma which shows that for each agent, starting at any given choice vector (for example, starting at any given initial portfolio), each arbitrage opportunity can be uniquely decomposed into the sum of two orthogonal net trade vectors: one vector specifying a trading direction in which the agent's utility is constant and one vector specifying a trading direction in which the agent's utility is nondecreasing. Thus, at each initial choice vector (i.e., at each starting point) there is for each agent a unique arbitrage coordinate system, determined by the direct sum of the subspace of useless or "utility-constant" net trades and its orthogonal complement of useful net trades. If the agent's utility-constant net trade subspace is the same at all choice vectors weakly preferred to the agent's endowment, then we say that an agent's preferences are weakly uniform. Under weak uniformity, a trading direction taken from the agent's utility-constant subspace at the agent's endowment is utility-constant no matter where the trading begins - as long as trading begins at a choice vector preferred to the endowment (i.e., uninteresting net trades are uniformly uninteresting).

With this arbitrage decomposition result in hand, we next examine the projections of the set of rational allocations upon agents' "utility-constant" net trade subspaces and their orthogonal complements. We show that Hart's (1974) condition of weak-no-market-arbitrage holds if and only if the projection of the set of rational allocations upon the Cartesian product of the agents' subspaces of useful net trades is compact. If in addition, agents' preferences are weakly uniform, then Hart's condition also implies compactness of the set of rational utility possibilities. Moreover, we show that if agents' utility-constant subspaces (at endowments) are linearly independent, then Hart's (1974) weak-no-market-arbitrage condition, Werner's (1987) no-arbitrage price condition, and Page's (1987) no-unbounded-arbitrage condition are equivalent, and in turn, all are equivalent to compactness of the set of rational allocations.

Using the geometry of arbitrage we sharpen and extend the result of Page, Wooders, and Monteiro (2000) showing the equivalence of the conditions of Hart (1974) and Werner (1987). In particular, we establish this equivalence without any assumptions concerning uniformity or nonsatiation. In Page, Wooders, and Monteiro (2000) the equivalence of Hart and Werner is obtained assuming a very weak form of nonsatiation (due to Werner (1987)) and a strong form of uniformity (i.e., uniformity of arbitrage opportunities). In addition, we show under weak uniformity only that the conditions of Hart and Werner imply the condition of inconsequential arbitrage, introduced in Page, Wooders, and Monteiro (2000). Page, Wooders, and Monteiro show this as well, but require Werner nonsatiation and strong uniformity. Finally, we show that inconsequential arbitrage implies compactness of the set of utility possibilities.

We also examine the problem of existence of equilibrium within the context of an unbounded exchange economy (i.e., an economy allowing unbounded short sales). The existence result of Dana, Le Van, and Magnien (1999) is central to our analysis. Throughout we maintain the classical assumption of *local* nonsatiation at rational allocations.[3] Under this nonsatiation assumption, Dana, Le Van, and Magnien (1999) show that compactness of utility possibilities is sufficient for the existence of an equilibrium. Their existence result together with our results concerning the relationship between some of the basic no-arbitrage conditions provide an overview of how these conditions fit together in the existence puzzle. In particular, we conclude that the stronger conditions of Hammond (1983) and Page (1987) imply existence, without uniformity conditions, by guaranteeing compactness of the set of rational allocation, while the weaker conditions of Hart (1974) and Werner (1987) require weak uniformity of preferences to guarantee compactness of utility possibilities, and therefore to guarantee existence via the Dana, Le Van, and Magnien (1999) result. Inconsequential arbitrage works differently. It implies compactness of the set of utility possibilities without any type of uniformity, and therefore, is sufficient for existence with-

[3]See Allouch, Le Van, and Page (2001) for an analysis of the existence problem with and without the assumption of nonsatiation on preferences.

out uniformity - again via the Dana, Le Van, and Magnien (1999) result.[4]

Finally, we present two results on necessary and sufficient conditions for the existence of equilibrium in an unbounded exchange economy. For our first result, we introduce the notion of weak no-half-lines - a weakening of Werner's no half line condition[5]. We then show that if the economy satisfies uniformity of arbitrage opportunities, local nonsatiation at rational allocation, and *weak* no-half-lines, then the Hart-Werner no-arbitrage conditions and inconsequential arbitrage are equivalent, and are necessary and sufficient for compactness of the set of utility possibilities and existence of equilibrium. Our second result is a corollary to our first: we show that if we strengthen the weak no-half lines condition to Werner's condition of no-half-lines, then the Hart-Werner no-arbitrage conditions and inconsequential arbitrage are equivalent to no-unbounded-arbitrage, and all are necessary and sufficient for compactness of the set of rational allocations, compactness of the set of utility possibilities, and existence of equilibrium.

The paper is organized as follows. In section 2, we present the basic elements of our model of an unbounded exchange economy. Also, we define arbitrage and present our basic results on the geometry of arbitrage. In section 3, we present our results on the relationship between the various no-arbitrage conditions found in the literature and the strength of the boundedness implied by these conditions. In section 4, we focus on sufficient conditions for existence of equilibrium, and in particular on the relationship between no-arbitrage conditions, implied boundedness, and the existence of equilibrium. Finally, in section 5, we present our results on conditions under which the various no-arbitrage conditions in the literature are equivalent and necessary and sufficient for existence of equilibrium.

3 The Model

We consider an economy $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ with m agents and l goods. Agent i has consumption set $X_i \subset R^l$, utility function $u_i(\cdot)$, and endowment $e_i \in X_i$. Agent $i's$ preferred set at $x_i \in X_i$ is

$$P_i(x_i) = \{x \in X_i \mid \; u_i(x) > u_i(x_i)\},$$

[4]However, in the case of the Hart and Werner conditions, existence can be established under nonsatiation assumptions considerably weaker than the local nonsatiation assumption made here (see Allouch, Le Van, and Page (2001)). As far as we know, this is not the case for inconsequential arbitrage.

[5]The weak no half line condition requires that a vector of net trades y be useless to agent i at x if the agent is indifferent along the half line $x+\lambda y$ for $\lambda \geq 0$. Thus, indifference along a half line implies indifference along the entire line (i.e., indifference along the line $x + \lambda y$ for $\lambda \in (-\infty, +\infty)$. Werner's no half line condition rules out indifference along half lines.

while the weak preferred set at x_i is

$$\widehat{P}_i(x_i) = \{x \in X_i \mid \; u_i(x) \geq u_i(x_i)\}.$$

The set of *individually rational allocations* is given by

$$\mathcal{A} = \{(x_i) \in \prod_{i=1}^{m} X_i \mid \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} e_i \text{ and } \; x_i \in \widehat{P}_i(e_i), \forall i\}.$$

We shall denote by $\mathcal{A}_i$ the projection of $\mathcal{A}$ onto X_i.

The set of *individually rational utility possibilities* is given by

$$\mathcal{U} = \{(v_i) \in R^m \mid \; \exists \; x \in \mathcal{A}, \text{ such that } u_i(e_i) \leq v_i \leq u_i(x_i), \; \forall i\}.$$

Definition 1 *(a) A rational allocation $x^* \in \mathcal{A}$ together with a* nonzero *vector of prices $p^* \in R^l$ is an equilibrium for the economy $\mathcal{E}$*

(i) if for each agent i and $x \in X_i$, $u_i(x) > u_i(x_i^*)$ implies $p^* \cdot x > p^* \cdot e_i$, and

(ii) if for each agent i, $p^* \cdot x_i^* = p^* \cdot e_i$.

(b) A rational allocation $x^* \in \mathcal{A}$ and a nonzero *price vector $p^* \in R^l$ is a quasi-equilibrium*

(i) if for each agent i and $x \in X_i$, $u_i(x) > u_i(x_i^*)$ implies $p^* \cdot x \geq p^* \cdot e_i$, and

(ii) if for each agent i, $p^* \cdot x_i^* = p^* \cdot e_i$.

Given (x^*, p^*) a quasi-equilibrium, it is well-known that if for each agent i, (i) $p^* \cdot x < p^* \cdot e_i$ for some $x \in X_i$ and (ii) $P_i(x_i^*)$ is relatively open in X_i, then (x^*, p^*) is an equilibrium. Conditions (i) and (ii) will be satisfied if, for example, for each agent i, $e_i \in \text{int}X_i$, and u_i is continuous on X_i. Using irreducibility assumptions, one can also show that a quasi-equilibrium is an equilibrium.

We now introduce our first two assumptions: for agents $i = 1, 2, \ldots, m$,

[A.1] X_i is closed and convex with $e_i \in X_i$,

[A.2] u_i is upper semicontinuous and quasi-concave.

Under these two assumptions, the weak preferred set $\widehat{P}_i(x_i)$ is convex and closed for $x_i \in X_i$.

3.1 The Geometry of Arbitrage

3.1.1 Definitions

We define the i^{th} *agent's arbitrage cone at* $x_i \in X_i$ as the closed convex cone containing the origin given by

$$O^+\widehat{P}_i(x_i) = \{y_i \in R^l \mid \forall x_i' \in \widehat{P}_i(x_i) \text{ and } \lambda \geq 0,\ x_i' + \lambda y_i \in \widehat{P}_i(x_i)\}.$$

Thus, if $y_i \in O^+\widehat{P}_i(x_i)$, then for all $\lambda \geq 0$ and all $x_i' \in \widehat{P}_i(x_i)$, $x_i' + \lambda y_i \in X_i$ and $u_i(x_i' + \lambda y_i) \geq u_i(x_i)$. The agent's arbitrage cone at x_i, then, is the recession cone corresponding the weak preferred set $\widehat{P}_i(x_i)$ (see Rockafellar (1970), Section 8).[6] Thus, if the agent, starting at x_i, trades in the $y_i \in O^+\widehat{P}_i(x_i)$ direction on any scale $\lambda \geq 0$, then his utility will never be less than $u_i(x_i)$. Put differently, if the agent, starting at x_i, trades in the $y_i \in O^+\widehat{P}_i(x_i)$ direction on any scale $\lambda \geq 0$, then there is no down-side utility (i.e., $u_i(x_i + \lambda y_i) \geq u_i(x_i)$ for all $\lambda \geq 0$). Given the definition of the arbitrage cone, we can give formal expression to an extended notion of potential arbitrage. In particular, we say that a set of net trades $y = (y_1, \ldots, y_m)$ is an arbitrage opportunity at $x = (x_1, \ldots, x_m)$ if

$\sum_{i=1}^m y_i = 0$ (i.e., trades are mutually compatible),
and
$y_i \in O^+\widehat{P}_i(x_i)$ for all i
(i.e., trades starting at x_i are without down-side utility).

A set closely related to the i^{th} agent's arbitrage cone is the *lineality space*, $L_i(x_i)$, of $\widehat{P}_i(x_i)$ given by

$$L_i(x_i) = \{y_i \in R^l \mid \ \forall x_i' \in \widehat{P}_i(x_i) \text{ and } \forall \lambda \in R,\ x_i' + \lambda y_i \in \widehat{P}_i(x_i)\}.$$

The set $L_i(x_i)$ consists of the zero vector and all the nonzero vectors y_i such that for each x_i' weakly preferred to x_i (i.e., $x_i' \in \widehat{P}_i(x_i)$), any vector z_i on the line through x_i' in the direction y_i, $z_i = x_i' + \lambda y_i$, is also weakly preferred to x_i (i.e., $z_i = x_i' + \lambda y_i \in \widehat{P}_i(x_i)$). The set $L_i(x_i)$ is a subspace of R^l, and is the largest subspace contained in the arbitrage cone $O^+\widehat{P}_i(x_i)$, (see Rockafellar (1970)). Moreover, since R^l is finite-dimensional, $L_i(x_i)$ is a *closed* subspace of R^l. Following the terminology of Werner (1987), we shall refer to net trades y_i contained in $L_i(x_i)$ as *useless* at x_i. If for all agents the subspace of *useless* net trades $L_i(x_i)$ is invariant with respect to starting point x_i for all x_i weakly preferred to the endowment e_i, then we say that the economy satisfies *weak uniformity.* We formalize this notion of uniformity in the following assumption:

[6] Equivalently, $y_i \in O^+\widehat{P}_i(x_i)$ if and only if y_i is a cluster point of some sequence $\{\lambda^k x_i^k\}_k$ where the sequence of positive numbers $\{\lambda^k\}_k$ is such that $\lambda^k \downarrow 0$, and where for all k, $x_i^k \in \widehat{P}_i(x_i)$; (see Rockafellar (1970), Theorem 8.2).

[A.3][Weak Uniformity] $L_i(x_i) = L_i(e_i), \forall x_i \in \widehat{P}_i(e_i), \forall i$.

For notational simplicity, we will denote each agent's arbitrage cone and lineality space at endowments in a special way. In particular, we will let

$$R_i := O^+\widehat{P}_i(e_i), \text{ and } L_i := L_i(e_i).$$

3.1.2 The Geometry of Arbitrage

In this subsection we present our main result on the geometry of arbitrage. In this result we utilize agents' subspaces of useless net trades and their orthogonal complements to expose the geometric structure common to all arbitrages. To begin, let $L_i^{\perp}(x_i)$ denote the space orthogonal to agent $i's$ subspace $L_i(x_i)$ of useless net trades at x_i. The vector space R^l can be decomposed into the direct sum of the lineality space $L_i(x_i)$ and its orthogonal complement, $L_i^{\perp}(x_i)$. Thus, for each $x_i \in X_i$,

$$R^l = L_i^{\perp}(x_i) \oplus L_i(x_i),$$

and thus, given $x_i \in X_i$ each vector $x \in R^l$ has a *unique* representation as the sum of two vectors, one from $L_i(x_i)$ and one from $L_i^{\perp}(x_i)$. Explicitly, for each $x \in R^l$, there exists uniquely $a \in L_i^{\perp}(x_i)$ and $b \in L_i(x_i)$, such that $x = a + b$.

Lemma 1 *Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying* [A.1]-[A.2]*. The following statements are true:*

1. *$\forall i, \forall x_i \in X_i$,*

   (a) $\widehat{P}_i(x_i) = (\widehat{P}_i(x_i) \cap L_i^{\perp}(x_i)) \oplus L_i(x_i)$,

   (b) $O^+\widehat{P}_i(x_i) = (O^+\widehat{P}_i(x_i) \cap L_i^{\perp}(x_i)) \oplus L_i(x_i)$.

2. *If in addition* [A.3] *holds (i.e., if weak uniformity holds), then*

$$u_i(x_i + y_i) = u_i(x_i), \forall x_i \in \widehat{P}_i(e_i) \text{ and } \forall y_i \in L_i.$$

3. *Let $\mathcal{A}^{\perp}$ be the projection of $\mathcal{A}$ onto $\prod_{i=1}^m L_i^{\perp}$. Then $\mathcal{A}^{\perp}$ is closed and convex.*

4. *Let $O^+(\mathcal{A})$ and $O^+(\mathcal{A}^{\perp})$ denote the recession cones of $\mathcal{A}$ and $\mathcal{A}^{\perp}$ respectively. Then*

$$O^+(\mathcal{A}^{\perp}) = \{(y_i) \in \prod_{i=1}^m (R_i \cap L_i^{\perp}) \mid \sum_{i=1}^m y_i \in \sum_{i=1}^m L_i\}.$$

5. *Let* $B = O^+(\mathcal{A}^\perp) + \prod_{i=1}^m L_i$. *Then*

$$O^+(\mathcal{A}) = \{(y_i) \in B \mid \sum_{i=1}^m y_i = 0\}.$$

Proof. (1) (a) Let $x_i' \in \widehat{P}_i(x_i)$. Since $R^l = L_i^\perp(x_i) \oplus L_i(x_i)$, x_i' is uniquely representable as $x_i' = y_i^\perp + \widehat{y}_i$ for some $y_i^\perp \in L_i^\perp(x_i)$ and $\widehat{y}_i \in L_i(x_i)$. Since $\widehat{y}_i \in L_i(x_i)$, $-\widehat{y}_i \in L_i(x_i)$. Thus, $y_i^\perp = y_i^\perp + \widehat{y}_i - \widehat{y}_i = x_i' - \widehat{y}_i \in \widehat{P}_i(x_i)$, and we have $y_i^\perp \in \widehat{P}_i(x_i) \cap L_i^\perp(x_i)$. Conversely, suppose $x_i' = y_i^\perp + \widehat{y}_i \in (\widehat{P}_i(x_i) \cap L_i^\perp(x_i)) \oplus L_i(x_i)$. Since $\widehat{y}_i \in L_i(x_i)$ and $y_i^\perp \in (\widehat{P}_i(x_i) \cap L_i^\perp(x_i)) \subset \widehat{P}_i(x_i)$, it follows that $x_i' = y_i^\perp + \widehat{y}_i \in \widehat{P}_i(x_i)$.

The proof of 1 (b) is similar.

(2) If the lineality space $L_i(x_i)$ is equal to the subspace L_i for all $x_i \in \widehat{P}_i(e_i)$ and if $y_i \in L_i$ then

$$u_i(x_i + y_i) \leq u_i(x_i + y_i - y_i) \leq u_i(x_i + y_i), \forall x_i \in \widehat{P}_i(e_i).$$

Therefore, $u_i(x_i + y_i) = u_i(x_i)$ for all $x_i \in \widehat{P}_i(e_i)$ and all $y_i \in L_i$.

(3) It is easy to verify that $\mathcal{A}^\perp$ is convex. Let us prove that $\mathcal{A}^\perp$ is closed. For any $x_i \in \widehat{P}_i(e_i)$, write $x_i = x_i^\perp + \widehat{x}_i$ for $x_i^\perp \in \widehat{P}_i(e_i) \cap L_i^\perp$ and $\widehat{x}_i \in L_i$. Let $\{(x_i^{\perp n})\}_n$ be a sequence in $\mathcal{A}^\perp$ such that $\lim_{n\to+\infty}(x_i^{\perp n}) = (x_i^\perp)$. For each n, there exists $(\widehat{x}_i^n) \in \prod_{i=1}^m L_i$, such that $\sum_{i=1}^m x_i^{\perp n} + \sum_{i=1}^m \widehat{x}_i^n = \sum_{i=1}^m e_i$. Hence, $\lim_{n\to+\infty} \sum_{i=1}^m \widehat{x}_i^n = \zeta \in \sum_{i=1}^m L_i$ since $\sum_{i=1}^m L_i$ is a finite dimensional subspace and hence closed. Now write $\zeta = \sum_{i=1}^m \zeta_i$, where for each i, $\zeta_i \in L_i$. One can easily check that for each i, $x_i^\perp \in \widehat{P}_i(e_i) \cap L_i^\perp$, and $(x_i^\perp + \zeta_i) \in \mathcal{A}$. Hence $(x_i^\perp) \in \mathcal{A}^\perp$.

(4) Let $(\mathbf{y}_i) \in O^+(A^\perp)$. We first prove that $(y_i) \in R_i \cap L_i^\perp$.

(i) $y_i \in L_i^\perp, \forall i$: Let $(x_i^\perp) \in A^\perp$. Then, for all $\lambda \geq 0$, we have $x_i^\perp + \lambda y_i \in L_i^\perp$. Hence, $y_i \in L_i^\perp$ since $L_i^\perp$ is a vector subspace and $x_i^\perp \in L_i^\perp$.

(ii) $y_i \in R_i$: Observe that if $(x_i^\perp) \in A^\perp$ then $u_i(x_i^\perp) \geq u_i(e_i), \forall i$. Indeed, there exists $(\widehat{x}_i) \in \prod_{i=1}^m L_i(e_i)$, such that $u_i(x_i^\perp + \widehat{x}_i) \geq u_i(e_i), \forall i$. Because $-\widehat{x}_i \in L_i \subset R_i$, we have $u_i(x_i^\perp) = u_i(x_i^\perp + \widehat{x}_i - \widehat{x}_i) \geq u_i(e_i)$.

Now, for any $\lambda \geq 0$, we have $(x_i^\perp + \lambda y_i) \in A^\perp$. Hence, $\forall i$ and $\forall \lambda \geq 0$, $u_i(x_i^\perp + \lambda y_i) \geq u_i(e_i)$. Since $x_i^\perp \in \widehat{P}_i(e_i)$, we have $y_i \in R_i$.

Next, we prove that $\sum_{i=1}^m y_i \in \sum_{i=1}^m L_i$.

Let $(y_i) \in O^+(\mathcal{A}^\perp)$ and $(x_i^\perp) \in \mathcal{A}^\perp$. For each integer n, there exists $(\widehat{x}_i^n) \in \prod_{i=1}^m L_i$, such that

$$\sum_{i=1}^m x_i^\perp + n\sum_{i=1}^m y_i + \sum_{i=1}^m \widehat{x}_i^n = \sum_{i=1}^m e_i.$$

Thus,

$$\lim_{n\to+\infty} \frac{\sum_{i=1}^m \widehat{x}_i^n}{n} = \zeta = \sum_{i=1}^m \zeta_i \text{ and } \zeta_i \in L_i, \forall i.$$

Since $\sum_{i=1}^m y_i + \sum_{i=1}^m \zeta_i = 0$, we have $\sum_{i=1}^m y_i \in \sum_{i=1}^m L_i$. Conversely, let $(y_i) \in \prod_{i=1}^m (R_i \cap L_i^\perp)$ such that $\sum_{i=1}^m y_i - \sum_{i=1}^m \zeta_i = 0, \zeta_i \in L_i, \forall i$. Let $(x_i^\perp + \widehat{x}_i) \in \mathcal{A}$, such that $x_i^\perp \in \mathcal{A}^\perp$. We have

$$((x_i^\perp + \lambda y_i) + (\widehat{x}_i - \lambda \zeta_i)) \in \mathcal{A}, \forall \lambda \geq 0.$$

Hence $(y_i) \in O^+(\mathcal{A}^\perp)$.

(5) Let us prove the formula for $O^+(\mathcal{A})$. It is obvious that

$$\begin{aligned} O^+(\mathcal{A}) &= \{(y_i) \in \prod_{i=1}^m R_i \mid \sum_{i=1}^m y_i = 0\} \\ &= \{(y_i^\perp + \widehat{y}_i) \mid (y_i^\perp, \widehat{y}_i) \in \prod_{i=1}^m ((R_i \cap L_i^\perp) \times L_i) \text{ and } \sum_{i=1}^m (y_i^\perp + \widehat{y}_i) = 0\} \\ &= \{(y_i) \in B \mid \sum_{i=1}^m y_i = 0\}, \text{by statement (4) above. } \blacksquare \end{aligned}$$

By Lemma 1(1)(a), each consumption vector $x \in X_i$ weakly preferred to consumption vector $x_i \in X_i$ has a unique representation as the sum of a purely useful and weakly preferred consumption vector $a_i \in \widehat{P}_i(x_i) \cap L_i^\perp(x_i)$ and a purely useless consumption vector $b_i \in L_i(x_i)$. Thus, for each $x \in \widehat{P}_i(x_i)$ there exists uniquely vectors $a_i \in \widehat{P}_i(x_i) \cap L_i^\perp(x_i)$ and $b_i \in L_i(x_i)$ such that

$$x = a_i + b_i.$$

Moreover, by Lemma 1(1)(b) each vector of net trades y contained in the arbitrage cone $O^+\widehat{P}_i(x_i)$ has a unique representation as the sum of a purely useful net trade vector $\alpha_i \in O^+\widehat{P}_i(x_i) \cap L_i^\perp(x_i)$ and a purely useless net trade vector $\beta_i \in L_i(x_i)$. Thus, for each $y \in O^+\widehat{P}_i(x_i)$ there exists uniquely vectors $\alpha_i \in O^+\widehat{P}_i(x_i) \cap L_i^\perp(x_i)$ and $\beta_i \in L_i(x_i)$ such that

$$y = \alpha_i + \beta_i.$$

Finally, by Lemma 1.2, under weak uniformity, [A.3], starting at any consumption vector $x_i \in X_i$ a useless net trade vector can be added or subtracted from x_i without changing utility.

4 No-Arbitrage Conditions and Compactness

4.1 Weak No Market Arbitrage

Hart (1974) introduced the *weak no-market-arbitrage* condition (WNMA). Hart's condition, a condition on net trades, requires that all mutually compatible arbitrage opportunities be useless. We have the following definition:

Definition 2 *The economy* $\mathcal{E}$ *satisfies the* WNMA *condition if*

$$\sum_{i=1}^{m} y_i = 0 \textit{ and } y_i \in R_i \textit{ for all } i, \textit{ then}$$
$$y_i \in L_i \textit{ for all } i.$$

Our next result tells us that Hart's condition is equivalent to the condition that $\mathcal{A}^{\perp}$ be compact. More importantly, it tells us that if the economy satisfies weak uniformity, then Hart's condition implies that the set of rational utility possibilities is compact.

Theorem 2 *Let* $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^{m}$ *be an economy satisfying* [A.1]-[A.2]. *The following statements are true:*

1. WNMA *holds if and only if* $\mathcal{A}^{\perp}$ *is compact. In this case,*

$$O^+(\mathcal{A}) = \{(y_i) \in \prod_{i=1}^{m} L_i \mid \sum_{i=1}^{m} y_i = 0\}.$$

2. *If in addition* [A.3] *holds (i.e., if weak uniformity holds), then if* $\mathcal{E}$ *satisfies* WNMA*, then the set of rational utility possibilities,* $\mathcal{U}$*,is compact.*

Proof. (1) It is obvious that

$$O^+(\mathcal{A}) = \{(y_i) \in \prod_{i=1}^{m} R_i \mid \sum_{i=1}^{m} y_i = 0\}.$$

Then, the WNMA condition holds if and only if

$$O^+(\mathcal{A}) = \{(y_i) \in \prod_{i=1}^{m} L_i \mid \sum_{i=1}^{m} y_i = 0\}.$$

From Lemma 1(5) we have

$$O^+(\mathcal{A}) = \{(y_i) \in O^+(\mathcal{A}^{\perp}) + \prod_{i=1}^{m} L_i \mid \sum_{i=1}^{m} y_i = 0\}.$$

Therefore, the WNMA condition holds if and only if $O^+(\mathcal{A}^{\perp}) = \{0\}$, which is equivalent to $\mathcal{A}^{\perp}$ being compact.

(2) If we add [A.3], then Lemma 1(2) implies that $\forall i, u_i(x_i) = u_i(x_i^{\perp}), \forall x_i \in \widehat{P}_i(e_i)$. Since $\mathcal{A}^{\perp}$ is compact, it follows that $\mathcal{U}$ is compact. ■

Theorem 2 is a direct consequence of Lemma 1. By part 1 of Theorem 2, the only potential arbitrage opportunities are those consisting of mutually compatible, useless net trades (i.e., WNMA holds) if and only if $\mathcal{A}^{\perp}$ is compact. Moreover, by part 2 of Theorem 2, under weak uniformity, [A.3], the set of individually rational utility possibilities is compact. In fact, we can write

$$\mathcal{U} = \{(v_i) \in R^m \mid \exists\, x^{\perp} \in \mathcal{A}^{\perp}, \text{such that } u_i(e_i) \leq v_i \leq u_i(x_i^{\perp}),\ \forall i\}.$$

Thus, under [A.2] (upper semicontinuity and quasi-concavity of utility functions), the compactness of $\mathcal{A}^{\perp}$ implies the compactness of $\mathcal{U}$.

4.2 No Unbounded Arbitrage

Page (1987) introduced the *no-unbounded-arbitrage* condition (NUBA). Page's condition, a condition on net trades stronger than Hart's, requires that all mutually compatible arbitrage opportunities be trivial. We have the following definition:

Definition 3 *The economy $\mathcal{E}$ satisfies the* NUBA *condition if*

$$\textstyle\sum_{i=1}^{m} y_i = 0 \textit{ and } y_i \in R_i \textit{ for all } i, \textit{ then} \\ y_i = 0 \textit{ for all } i.$$

Our next result tells us that Page's condition is equivalent to the condition that $\mathcal{A}$ be compact. More importantly, it tells us that if agents' lineality spaces are linearly independent, then Hart's condition and Page's condition are equivalent. This latter result extends Proposition 5.1 in Page (1987).

Theorem 3 *Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying* [A.1]-[A.2]. *The following statements are equivalent:*

1. *$\mathcal{E}$ satisfies* NUBA.
2. *$\mathcal{A}$ is compact.*
3. *$\mathcal{A}^{\perp}$ is compact and the lineality spaces, L_i, are linearly independent.*
4. *$\mathcal{E}$ satisfies* WNMA *and the lineality spaces, L_i, are linearly independent.*

Proof. (1) $\Leftrightarrow$ (2) : It is obvious that,

$$O^+(\mathcal{A}) = \{(y_i) \in \prod_{i=1}^{m} R_i \mid \sum_{i=1}^{m} y_i = 0\}.$$

Hence, the NUBA condition is satisfied if and only if $O^+(\mathcal{A}) = \{0\}$, which is equivalent to $\mathcal{A}$ is compact.

(2) $\Leftrightarrow$ (3) : From Lemma 1(5) we have

$$O^+(\mathcal{A}) = \{(y_i) \in O^+(\mathcal{A}^{\perp}) + \prod_{i=1}^{m} L_i \mid \sum_{i=1}^{m} y_i = 0\}.$$

Clearly, if $O^+(\mathcal{A}^{\perp}) = \{0\}$ and the lineality spaces (L_i) are linearly independent then $O^+(\mathcal{A}) = \{0\}$. Conversely, suppose that $O^+(\mathcal{A}) = \{0\}$. It is clear that (L_i) are linearly independent, since $L_i \subset R_i$. Moreover, let $(y_i) \in O^+(\mathcal{A}^{\perp})$. From Lemma 1 (4) there exists $(\zeta_i) \in \prod_{i=1}^{m} L_i$ such that

$\sum_{i=1}^{m} y_i - \sum_{i=1}^{m} \zeta_i = 0$. By Lemma 1(5), $(y - \zeta) \in O^+(A)$. Since $O^+(A) = \{0\}$, we have $y = \zeta$. Hence $y = 0$, since $y_i \in L_i^{\perp}$, for all i.

(3) $\Leftrightarrow$ (4) : Follows directly from Theorem 2 above. ■

Remark Note that if $\mathcal{E}$ satisfies NUBA, then the set of rational utility possibilities, $\mathcal{U}$, is compact.

Lemma 1 is useful in interpreting NUBA and in comparing NUBA to WNMA. In particular, under WNMA all arbitrage opportunities must occur in the lineality spaces, L_i. NUBA is stronger and requires that there be no arbitrage opportunities in either the lineality spaces, L_i, or their orthogonal compliments, $L_i^{\perp}$.

4.3 No Arbitrage Price System

Werner (1987) introduced the *no-arbitrage price system* condition (NAPS). Werner's condition, a condition on prices, requires that there be a nonempty set of prices such that each price contained in this nonempty subset assigns a positive value to any vector of *useful* net trades belonging to any agent. Werner then assumes that for each agent the set of useful net trades at endowments is nonempty. In particular, Werner assumes that

[WNS] [Werner nonsatiation] $R_i \setminus L_i \neq \emptyset$, $\forall i$.

We have the following definition:

Definition 4 *In an economy $\mathcal{E}$ satisfying* [WNS]*, the* NAPS *condition is satisfied if*

$$\bigcap_{i=1}^{m} S_i^W \neq \emptyset,$$

where

$$S_i^W = \{p \in R^\ell \mid \ p \cdot y > 0, \forall y \in R_i \setminus L_i\}$$

is Werner's cone of no-arbitrage prices.

Here, we shall extend Werner's condition to allow for the possibility that for some agent the set of useful net trades is empty - that is, to allow for the possibility that for some agent, $R_i = L_i$. More importantly, we shall prove, under very mild conditions, that our extended version of Werner's condition is equivalent to Hart's condition. This result extends an earlier result by Page, Wooders, and Monteiro (2000) on the equivalence of Hart and Werner conditions. We begin by extending the definition of Werner's cone of *no-arbitrage prices*:

Definition 5 *For each agent i, define*

$$S_i = \begin{cases} S_i^W & \text{if } R_i \setminus L_i \neq \emptyset, \\ L_i^{\perp} & \text{if } R_i = L_i. \end{cases}$$

Given this expanded definition of no-arbitrage-price cone, the extended NAPS condition is defined as follows:

Definition 6 *The economy $\mathcal{E}$ satisfies the* NAPS *condition if*

$$\bigcap_{i=1}^{m} S_i \neq \emptyset.$$

Remark Note that if the economy $\mathcal{E}$ satisfies Werner's nonsatiation condition, i.e., $R_i \setminus L_i \neq \emptyset$, $\forall i$, then the NAPS condition given in Definition 6 above reduces to Werner's original condition given in Definition 4.

In order to prove the equivalence of NAPS and WNMA, we need the following Lemma.

Lemma 4 *Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying* [A.1]-[A.2]. *The following statements are true:*

1. For any i, such that $R_i \setminus L_i \neq \emptyset$, we have:

$$S_i = \{p \in L_i^{\perp} \mid\ p.y > 0, \forall y \in (R_i \cap L_i^{\perp}) \setminus \{0\}\}.$$

2. $\forall i = 1, \dots, m, S_i = -\text{ri}(R_i^0)$ where R_i^0 is the polar cone of R_i.

Proof. (1) See Dana, Le Van and Magnien (1999, p.182).

(2) It is clear that if $R_i = L_i$ then $R_i^0 = L_i^{\perp} = S_i$. Thus, $S_i = \text{ri}(-{R_i}^0)$. Now let us suppose that $R_i \setminus L_i \neq \emptyset$. First, we show that $\text{aff}(R_i^0) = L_i^{\perp}$. Indeed, since $L_i \subset R_i$ we have $R_i^0 \subset L_i^{\perp}$ and then $\text{aff}(R_i^0) \subset L_i^{\perp}$. Furthermore, if $\text{aff}(R_i^0)$ is a proper vector subspace of $L_i^{\perp}$, then L_i is a proper vector subspace of $(\text{aff}(R_i^0))^{\perp}$. But $(\text{aff}(R_i^0))^{\perp} \subset R_i$, which contradicts the fact that the lineality space L_i is the maximal vector subspace contained in R_i.

From Lemma 1(1) we have $R_i = (R_i \cap L_i^{\perp}) + L_i$. Corollary 16.4.2 in Rockafellar (1970) gives

$$\begin{aligned} R_i^0 &= (R_i \cap L_i^{\perp})^0 \cap L_i^{\perp} && (1) \\ &= \{p \in L_i^{\perp} \mid\ p.y \leq 0, \forall y \in (R_i \cap L_i^{\perp})\} && (2) \end{aligned}$$

We notice that the positive dual of $R_i \cap L_i^{\perp}$ in $L_i^{\perp}$ is also R_i^0, and that $R_i \cap L_i^{\perp}$ is pointed cone, that is:

$$(R_i \cap L_i^{\perp}) \bigcap -(R_i \cap L_i^{\perp}) = 0$$

Then, it follows from (2)

$$\mathrm{ri}R_i^0 = \mathrm{int}_{L_i^\perp} R_i^0 = \{p \in L_i^\perp \mid \ p \cdot y < 0, \forall y \in (R_i \cap L_i^\perp) \setminus \{0\}\}.$$

From (1) of the present lemma, we get $S_i = -\mathrm{ri}(R_i^0)$. ■

By Lemma 4(1) a no-arbitrage price vector is a price vector for which the value of any useful net trade vector in the arbitrage cone is positive while the value of any useless net trade vector is zero.

Werner (1987) assumes that each agent's arbitrage cone is invariant with respect to the starting point of the trading (i.e., x_i), as long as the starting point is weakly preferred to the agent's endowment (i.e., as long as, $x_i \in \widehat{P}_i(e_i)$). That is, Werner assumes:

[A$'$.3][Uniformity] $O^+\widehat{P}_i(x_i) = R_i$, $\forall x_i \in \widehat{P}_i(e_i), \forall i$.

Note that if uniformity [A$'$.3] holds, then weak uniformity [A.3] holds automatically. That is [A$'$.3] implies that $L_i(x_i) = L_i, \forall x_i \in \widehat{P}_i(e_i), \forall i$.

Page, Wooders and Monteiro (2000) show that under [A.1]-[A.2], [A$'$.3] and [WNS], WNMA holds if and only if $\bigcap_{i=1}^m S_i^W \neq \emptyset$ (i.e., Hart's condition holds if and only if Werner's condition holds). Here, we extend this result by proving, under [A.1]-[A.2] only, that WNMA holds if and only if $\bigcap_{i=1}^m S_i \neq \emptyset$.

To prove this statement, in addition to Lemma 4 above, we need the following lemma, a restatement of Corollary 16.2.2 in Rockafellar (1970).

Lemma 5 *Let $f_1, \ldots f_m$ be a proper convex functions on R^m. In order that there do not exist vectors $x_1^*, \ldots, x_m^*$ such that*

$$x_1^* + \ldots + x_m^* = 0 \tag{3}$$

$$f_1^*O^+(x_1^*) + \ldots + f_m^*O^+(x_m^*) \leq 0, \tag{4}$$

$$f_1^*O^+(-x_1^*) + \ldots + f_m^*O^+(-x_m^*) > 0, \tag{5}$$

it is necessary and sufficient that

$$\bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom} f_i) \neq \emptyset.$$

We recall that for a convex function $\mathrm{dom} f_i = \{x \in R^m \mid \ f_i(x) < +\infty\}$ and $f_i^*O^+$ is the support function of $\mathrm{dom} f_i$, that is,

$$f_i^*O^+(x_i^*) = \sup\{x_i^* \cdot x \mid \ x \in \mathrm{dom} f_i\}.$$

Theorem 6 *Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying* [A.1]-[A.2]. *The following statements are equivalent:*

1. $\mathcal{E}$ *satisfies* WNMA.

2. $\mathcal{E}$ *satisfies* NAPS.

Proof. For every $i = 1, \dots, m$, let

$$f_i(x) = \begin{cases} 0 & \text{if } x \in R_i^0, \\ +\infty & \text{otherwise.} \end{cases}$$

Hence

$$f_i^* O^+(x_i^*) = \sup\{x_i^* \cdot x \mid \; x \in R_i^0\}. \tag{6}$$

Since $0 \in R_i^0$, it follows that $f_i^* O^+(x_i^*) \geq 0$ for all i. Then (4) is satisfied if and only if $f_i^* O^+(x_i^*) = 0$ for all i and therefore from (6) if and only if $x_i^* \in R_i$. Quite similarly, (5) is not satisfied if and only if $-x_i^* \in R_i$. Since $L_i = R_i \cap -R_i$, it follows that the first assertion of Lemma 5 is satisfied if and only if the WNMA condition is satisfied. Furthermore, from Lemma 4 one gets

$$\bigcap_{i=1}^{m} S_i = \bigcap_{i=1}^{m} \mathrm{ri}(-{R_i}^0) = -\bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom} f_i).$$

Hence, the equivalence follows from Lemma 5. ■

Page and Wooders (1996) state that if $L_i = \{0\}, \forall i$, then NUBA holds if and only if $\bigcap_{i=1}^m S_i^W \neq \emptyset$. In fact, this result is a consequence of a sharper result:

Corollary 7 *Let* $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ *be an economy satisfying* [A.1]-[A.2]. *The following statements are equivalent:*

1. $\mathcal{E}$ *satisfies* NUBA.

2. $\bigcap_{i=1}^m S_i \neq \emptyset$ *and the lineality spaces are linearly independent.*

Proof. It follows from Theorem 3 (1) and Theorem 6. ■

Remark By the Corollary 7 there is an absence of arbitrage opportunities if and only if there exists a price system limiting arbitrage opportunities contained in the $L_i^\perp$ spaces and there are no arbitrage opportunities in the lineality spaces. Thus, when the lineality spaces are equal to zero, nonemptiness of the set of no-arbitrage prices (i.e., $\bigcap_{i=1}^m S_i \neq \emptyset$) is necessary and sufficient to rule out arbitrage opportunities in the economy.

4.4 Inconsequential Arbitrage

Page, Wooders and Monteiro (2000) extend the Hart (1974) model to an abstract general equilibrium setting without uniformity conditions and introduce a condition limiting arbitrage, called *inconsequential arbitrage* (IC). Their condition is weaker than the weak no-market-arbitrage condition and implies compactness of the utility set, $\mathcal{U}$.

A set of net trades $y = (y_1, \ldots, y_m) \in R^{\ell m}$ is an *arbitrage* in economy $\mathcal{E}$ if y is the limit of some sequence $\{\lambda^n x^n\}_n$ where $\lambda^n \downarrow 0$ and $\{x^n\}_n \subseteq \mathcal{A}$ is a sequence of rational allocations. They denote the set of all arbitrages by

$$\text{arb}(\mathcal{E}) = \{y \in R^{\ell m} \mid \exists\ \{x^n\}_n \subseteq \mathcal{A} \text{ and } \lambda^n \downarrow 0 \text{ such that } y = \lim_{n\to+\infty} \lambda^n x^n\},$$

and they denote by

$$\text{arbseq}(y) = \left\{\{x^n\}_n \subseteq \mathcal{A} \mid \exists\ \lambda^n \downarrow 0 \text{ such that } y = \lim_{n\to+\infty} \lambda^n x^n\right\},$$

the set of all *arbitrage sequences* corresponding to $y \in \text{arb}(\mathcal{E})$.

Definition 7 *The economy $\mathcal{E}$ satisfies the* (IC) *condition if for all $y \in \text{arb}(\mathcal{E})$ and $\{x^n\}_n \in \text{arbseq}(y)$, there exists an $\epsilon > 0$ such that for all n sufficiently large*

$$x_i^n - \epsilon y_i \in X_i \textit{ and } u_i(x_i^n - \epsilon y_i) \geq u_i(x_i^n), \forall i.$$

Theorem 8 *Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying* [A.1]-[A.2]. *The following statements are true:*

1. NUBA *holds* $\Rightarrow$ IC *holds.*
2. *If, in addition,* [A3] *holds, then* WNMA *holds* $\Rightarrow$ IC *holds.*
3. IC *holds* $\Rightarrow \mathcal{U}$ *is compact.*

Proof. (1) It is clear since $O^+(\mathcal{A}) = \{0\}$.
(2) Let $y \in \text{arb}(\mathcal{E})$, then it follows from the WNMA condition that for each agent i, $y_i \in L_i$. It follows from Lemma 1(3) that for all $\{x^n\}_n \in \text{arbseq}(y)$ and $\epsilon > 0$,

$$x_i^n - \epsilon y_i \in X_i \text{ and } u_i(x_i^n - \epsilon y_i) = u_i(x_i^n), \forall i,$$

which ends the proof.
(3) See Allouch (1999), Page, Wooders, and Monteiro (2000). ■

5 Sufficient Conditions for the Existence of Equilibrium

In this section, we examine the relationship between the no-arbitrage conditions we have discussed and existence of equilibrium. We stated in the introduction that, in economic models of exchange economies allowing short sales, these conditions guarantee existence by endogenously bounding the economy. But as we have seen in Theorems 2, 3, and 8 above, this endogenous bounding takes the form of compactness of the set of rational utility possibilities. Thus in this section we begin by stating a fundamental existence result due to Dana, Le Van, and Magnien (1999). This result states that compactness of rational utility possibilities is sufficient for the existence of a quasi-equilibrium. Before we state the Dana, Le Van, and Magnien result we add to our list the following assumptions:

[A.4] [Local Nonsatiation] $\forall\ i$, $\forall x_i \in \mathcal{A}_i$, $\exists\ \{y_i^n\}_n \subset X_i$ with $\lim_{n\to+\infty} y_i^n = x_i$ and $u_i(y_i^n) > u_i(x_i), \forall n$.

[A.5] $\forall\ i$, $e_i \in \text{int}X_i$ and $\forall x_i \in \mathcal{A}_i$, $P_i(x_i)$ is relatively open in X_i.

Assumption [A.4] is standard. Assumption [A.5] allows us to conclude that a quasi-equilibrium for the economy is in fact an equilibrium for the economy. We now state the Dana, Le Van, and Magnien (1999) existence result.

Theorem 9 *(Compactness of the utility set is sufficient for existence)*
Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying assumptions [A.1], [A.2], and [A.4]. *If the set of rational utility possibilities $\mathcal{U}$ is compact, then $\mathcal{E}$ has a quasi-equilibrium. Moreover, if* [A.5] *holds, then $\mathcal{E}$ has an equilibrium.*

Putting together Theorem 9 and our Theorems 2, 3, 6, and 8, we can summarize the relationship between the no-arbitrage conditions that we have discussed and existence of equilibrium as follows:

Theorem 10 *(No-arbitrage conditions implying existence)*
Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying assumptions [A.1], [A.2], and [A.4]. *The following statements are true:*

1. *If* IC *holds, then $\mathcal{E}$ has a quasi-equilibrium.*
2. *If in addition the economy satisfies* [A.3]*, weak uniformity, then*
   (a) if WNMA *holds, then $\mathcal{E}$ has a quasi-equilibrium,*
   (b) if NAPS *holds, then $\mathcal{E}$ has a quasi-equilibrium.*

3. If NUBA *holds, then $\mathcal{E}$ has a quasi-equilibrium.*

Proof. (1) is an immediate consequence of Theorems 8 and 9 above.

(2)(a) is an immediate consequence of Theorems 2(2) and 9 above.

(2) (b) is an immediate consequence of Theorems 6, 2(2), and 9 above. ■

Remark Part 2(b) of our Theorem improves upon the existence result of Werner's in the following sense. We show that an extended version of Werner's no-arbitrage price condition is sufficient for existence under weak uniformity [A.3]. Werner in his proof of existence requires the stronger condition of uniformity [A′.3]. However, Werner makes a different assumption concerning nonsatiation. In particular, Werner assumes [WNS] rather than local nonsatiation as we do here. In Allouch, Le Van, and Page (2001), we investigate the relationship between existence and nonsatiation using our extended version of Werner's no-arbitrage-price system condition. Part 2(a) improves upon the existence result of Hart. In particular, we extend Hart's condition to an abstract general equilibrium model and show that Hart's condition is sufficient for existence under weak uniformity [A.3]. Like Werner, Hart in his proof of existence requires that the stronger condition of uniformity [A′.3] hold.

6 Necessary and Sufficient Conditions for Existence of Equilibrium

In this our last section, we show that if the economy satisfies the additional condition of weak no-half-lines, then the conclusions of Theorem 10 can be greatly strengthened. In particular, under weak no-half-lines the conditions of Hart and Werner and inconsequential arbitrage are equivalent, and all are equivalent to the compactness of the set of rational utility possibilities and the existence of equilibrium. Stated formally, the weak no-half-lines condition is as follows:

[A.6] [Weak No-Half-Lines] $\forall x_i \in \widehat{P}_i(e_i)$, if $y \in R^l$, satisfies $u_i(x_i + \lambda y) = u_i(x_i), \forall \lambda \geq 0$, then $y \in L_i$.

If the economy satisfies uniformity [A′.3] as well as weak no-half-lines, then any potential arbitrage (i.e., any net trade vector contained in any agent's arbitrage cone) is either a direction in which the agent's utility is eventually increasing or a direction in which the agent's utility is constant.[7]

In order to prove our main equivalence result, we shall need the following lemma:

[7] An agent's utility is eventually increasing at x_i in direction y_i if given any $\lambda \geq 0$, there exists a $\lambda' > \lambda$ such that $u_i(x_i + \lambda' y_i) > u_i(x_i + \lambda y_i)$.

Lemma 11 *Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying assumptions*[A.1]-[A.2]*,* [A$'$.3] *(uniformity),* [A.4] *(local nonsatiation), and* [A.6] *(weak no-half-lines). Then any equilibrium price is a no-arbitrage price.*

Proof. Let (x^*, p^*) be an equilibrium.

first case. Assume $R_i \setminus L_i \neq \emptyset$. Take $y \in R_i \setminus L_i$. Then [A$'$.3] implies that $u_i(x_i^* + \lambda y) \geq u_i(x_i^*), \forall \lambda \geq 0$. But [A.6] implies there exists some $\lambda_0 > 0$ such that $u_i(x_i^* + \lambda_0 y) > u_i(x_i^*)$. Hence $p^*.y > 0$. Thus $p^* \in S_i$.

second case. Assume $R_i = L_i$. By [A.4], there exists $x_i \in X_i$ such that $u_i(x_i) > u_i(x_i^*)$. Then $\forall \lambda \in R \setminus \{0\}, \forall q \in L_i$, $u_i(x_i + \lambda q) = u_i(x_i) > u_i(x_i^*)$. Hence $p^*.(x_i + \lambda q) > p^*.e_i$. Divide by λ and let λ go to $+\infty$, we obtain $p^*.q = 0$. Thus, $p^* \in L_i^{\perp}$.

Thus, we have proved that $p^* \in \bigcap_{i=1}^m S_i$. ■

Now we have our main equivalence result.

Theorem 12 *Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying assumptions* [A.1]-[A.2]*,* [A$'$.3] *(uniformity),* [A.4] *(local nonsatiation),* [A.5]*, and* [A.6] *(weak no-half-lines). Then the following statements are equivalent:*

1. *$\mathcal{E}$ satisfies the no-arbitrage price system condition (Werner (1987)).*
2. *$\mathcal{E}$ satisfies the weak-no-market-arbitrage condition (Hart (1974)).*
3. *$\mathcal{E}$ satisfies inconsequential arbitrage (Page, Wooders, Monteiro (2000)).*
4. *The set of rational utility possibilities, U, is compact.*
5. *$\mathcal{E}$ has an equilibrium.*

Proof. $(1) \Leftrightarrow (2)$: Theorem 6.

$(2) \Rightarrow (3)$: Theorem 8(2).

$(3) \Rightarrow (4)$: Theorem 8(3).

$(4) \Rightarrow (5)$: Theorem 9.

$(5) \Rightarrow (1)$: Lemma 11. ■.

Our last result, a corollary to Theorem 12, shows that if we strengthen the weak the no-half-lines condition then no-unbounded-arbitrage (NUBA) and compactness of the set of rational allocations can be added to our list of equivalences. The strengthening of weak no-half -lines we shall assume is the following:

[A$'$.6] [No Half Line] $\forall x_i \in \widehat{P}_i(e_i)$, if $y \in R^l$, satisfies $u_i(x_i + \lambda y) = u_i(x_i), \forall \lambda \geq 0$, then $y = 0$.

Assumption [A$'$.6] is the *no-half-lines* assumption of Werner (1987). Note that assumption [A$'$.6] implies [A.6].

Corollary 13 *Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying assumptions* [A.1]-[A.2], [A$'$.3] *(uniformity),* [A.4] *(local nonsatiation),* [A.5], *and* [A$'$.6] *(no-half-lines). Then the following statements are equivalent:*

1. *$\mathcal{E}$ satisfies the no-arbitrage price system condition (Werner (1987)).*
2. *$\mathcal{E}$ satisfies the weak-no-market-arbitrage condition (Hart (1974)).*
3. *$\mathcal{E}$ satisfies the no-unbounded-arbitrage condition (Page (1987)).*
4. *The set of rational allocations, $\mathcal{A}$, is compact.*
5. *$\mathcal{E}$ satisfies inconsequential arbitrage (Page, Wooders, Monteiro (2000)).*
6. *The set of rational utility possibilities, U, is compact.*
7. *$\mathcal{E}$ has an equilibrium.*

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