Catalog Competition and Stable Nonlinear Prices

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Dedicated to Roko Aliprantis on his 60^{th} birthday

Abstract

We model strategic competition between product differentiated oligopolists in a market with privately informed buyers as an abstract game over market situations. In this game each firm's strategy space consists of a set of catalogs - and each catalog in turn consists of a set of products and prices the firm might offer to the market. Assuming that firms behave farsightedly in choosing their catalog strategies, we specify the market situation game by two objects: (i) a set of market situations, that is, a set of feasible profit-catalog profiles for firms, and (ii) a dominance relation defined on the set of market situations which reflects farsighted behavior. We show that the set of market situations is compact and we introduce two dominance relations on the set of market situations: farsighted dominance and path dominance. We then identify conditions sufficient to guarantee the existence of a nonempty set of market situations stable with respect to farsighted dominance (i.e., a nonempty largest farsightedly consistent set), as well as conditions sufficient to guarantee the existence of a nonempty set of market situations stable with respect to path dominance. Finally, we show that for any finite market situation game there exists a stable set with respect to path dominance contained in largest farsightedly consistent set. We close with an example illustrating this relationship between path dominance stability and farsighted consistency for finite market situation games.

KEYWORDS: competitive nonlinear pricing, delegation principle, catalog games, farsighted stability. JEL Classifications: C6, C7, D4

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1 Introduction

Firms often compete by offering potential buyers catalogs of products and prices. Examples of catalog competition abound. Fidelity Investments Inc. competes in the mutual funds market by offering investors a catalog of funds with differing risks and fee structures (e.g., equity funds, bond funds, index funds). California competes with other states to attract businesses by offering a catalog of amenities and taxes. Intel competes with other high tech firms to attract top engineers by offering a catalog of compensation and benefit packages. Wal Mart competes with other discount chains by offering consumers a catalog of store locations, products, and prices. And the list could go on. All of these examples have two features in common: multidimensional competition and asymmetric information. In particular, firms compete simultaneously in prices and products (broadly defined) and firms do not know all of the relevant characteristics of potential buyers. By offering a catalog firms are better able to screen potential buyers by allowing buyers to sort themselves. Moreover, by offering a well-chosen catalog of products and prices - including products not often chosen by customer types in the firm's customer base - firms are better able to compete multidimensionally and often times are better able to deter a competitor from defecting to a new catalog or to prevent such a defection from eroding the firm's customer base.

This paper has three main objectives: (1) to develop a game-theoretic model of catalog competition in markets with privately informed buyers where firms behave farsightedly in choosing their catalog strategies, (2) to develop notions of stability which reflect this farsighted strategic behavior, and (3) to identify conditions sufficient to guarantee the existence of market situations which are stable with respect to farsighted behavior.

In order to accomplish our first objective, we introduce the notion of an abstract market situation game (an abstract game in the sense of von Neumann-Morgenstern (1944)). This game consists of two objects: (i) a set of market situations, that is, a set of feasible profit-catalog profiles for firms, and (ii) a farsighted dominance relation defined on the set of market situations. A status quo market situation is said to be farsightedly dominated by an alternative market situation if there is a finite sequence of changes in the status quo leading to the alternative situation such that in the alternative market situation firms who initiated the change (via changes in their catalogs), as well as the firms who initiated changes in the temporary status quo they faced along the way, are made better off. By considering a market situation game equipped with the farsighted dominance relation we provide one possible way of addressing the problem of *myopia* expressed in Fisher's (1898) criticism of Cournot's duopoly model:

> "*No business man assumes that either his rival's output or price will remain constant... On the contrary, his whole thought is to forecast what move the rival will make in response to one of his own.*"[1]

[1] This quote from Fisher (1898) also appears in the introduction of Michael Chwe's elegant 1994 paper in the *Journal of Economic Theory.*

To accomplish our second objective (i.e., developing stability notions which reflect farsighted behavior), we begin by applying the notion of *farsighted consistency* introduced by Chwe (1994). Stated informally, a set of market situations is said to be consistent with respect to farsighted dominance if at each market situation contained in the set, any deviation to an alternative market situation carries with it the possibility that a sequence of further deviations might occur which in the end leaves at least one of the initially deviating firms not better off - and possibly worse off. Thus, if we assume that firms are farsightedly conservative, that is, if we assume that in any status quo market situation firms are deterred from defecting to alternative situations (via changes in their catalogs) by even the possibility of further unfavorable defections, then the largest set of farsightedly consistent market situations will contain all market situations which are conservatively stable with respect to farsighted dominance.

In order to establish our results on the existence of farsightedly consistent market situations - and thus the existence of farsightedly stable catalog profiles - we begin by showing that under very mild conditions on primitives the set of all market situations is compact. Given this compactness result we are able to conclude that, in general, arbitrarily close to any market situation game there is a finite market situation subgame. Then, applying Chwe (1994), we show that each such finite approximating subgame has a unique, nonempty largest consistent set of market situations - and thus, a nonempty farsightedly stable set of catalog profiles. As a corollary, we conclude that if the set of market situations is finite, then the game has a unique, nonempty largest consistent set.

Next, we equip the set of market situations with a different type of farsighted dominance relation, namely, the path domination relation (i.e., the PD relation) introduced in Page and Wooders (2004) for network formation games. A status quo market situation is said to be path dominated by an alternative market situation if there is a *finite* sequence of market situations, beginning with the status quo and ending with the alternative, such that each market situation along the path is farsighted dominated by its successor. We call such a finite sequence of market situations a finite domination path. Applying classical results from graph theory due to Berge (2001), we conclude that if the market situation game equipped with the path dominance relation (i.e., the PD market situation game) is *inductive*, then it possesses a von Neumann-Morgenstern stable set with respect to path dominance (i.e., a PD-stable set), and if a PD market situation game possesses a *finite* PD-stable set, then it is inductive.[2] A PD market situation game is said to be inductive if given any domination path (finite or infinite) there exists a market situation which path dominates each market situation along the path. Thus, all inductive PD market situation games possess a nonempty set of PD-stable catalog profiles. Since all finite PD market situation games are automatically inductive, as a corollary, we conclude that if the set

[2] A set of market situations is said to be a von Neumann-Morgenstern stable set with respect to the path dominance, if given any two distinct situations in the set there does not exist a finite domination path connecting the two situations (internal path stability) and if for any market situation not contained in the set, there does exist a finite domination path from this situation to some situation in the set (external path stability).

of market situations is finite (e.g., if the set of all possible products and prices is finite), then the game has a von Neumann-Morgenstern stable set with respect to path dominance.

We close our discussion of path dominance stability by noting that it follows from Page and Wooders (2004) that all finite PD market situation games possess a von Neumann-Morgenstern PD-stable set which is contained in the largest consistent set of the corresponding market situation game (i.e., the game with respect to farsighted dominance). We illustrate this result with an example.

2 The Catalog Model of Strategic Competition in Markets with Privately Informed Buyers

We construct a model in which m firms, indexed by i *and* j ($= 1, 2, \ldots, m$), compete for the business of a single, privately informed agent via the catalogs of products and prices firms offer to the agent. The agent represents for each of his types a buyer in the market. In our model, firms move first, choosing their catalogs. The agent moves second, choosing for each of his types a firm with which to do business and a product-price pair from that firm's catalog. The market situation game we shall ultimately consider provides a model of how firms choose their catalogs in the first stage given the distribution of agent types (i.e., the distribution of buyers) and the optimizing behavior of buyers in the second stage. We begin by considering the second stage, that is, by considering the agent's problem.

2.1 The Agent's Problem

2.1.1 Agent Types and Contracts

We shall assume that

(A-1) the set of agent types is given by a probability space, $(T, \Im, \mu)$, where T is a set of agent types, $\Im$ is a σ -field in T, and μ is a probability measure defined on $\Im$.

Under (A-1), multidimensional type descriptions are allowed.

Let X be a set representing all possible products firms can offer the agent and let D be a subset of the real numbers R representing the prices firms might charge. For each firm $i = 1, 2, \ldots, m$ let

$$K_i := X_i \times D.$$

be the i^{th} firm's feasible set of product-price pairs where $X_i \subseteq X$ is the set of all possible products firm i can offer to the market. Elements of K_i, denoted by (x, p), can be viewed as describing the relevant characteristics of the sales contracts offered by firm i. For example, if X_i is a subset of R^L (i.e., if $X_i \subseteq X \subseteq R^L$), then in sales contract $(x, p) \in K_i$, $x = (x_1, \ldots, x_L) \in X_i$ might describe product characteristics such as quantity, quality, and location, while $p \in D$ gives the price.

We shall assume that,

(A-2) (i) X is a compact metric space, containing an element 0 which we shall agree denotes "no contracting," (ii) X_i is a nonempty closed subset of X containing 0, and (iii) D is a closed bounded interval of the nonnegative reals containing 0.

In order to allow for the possibility that some types of the agent (i.e., some buyers) may wish to abstain from contracting altogether, we assume that there is a fictitious firm $i = 0$ with feasible set of products and prices $K_0 := \{(0,0)\}$. Thus, for fictitious firm 0, $P_f(K_0) = \{(0,0)\}$.

Letting $I = \{0, 1, 2, \ldots, m\}$, define the set

$$\mathbb{K} := \{(i, x, p) \in I \times X \times D : (x, p) \in K_i\}. \tag{1}$$

A firm-contract pair $(i, x, p) \in \mathbb{K}$ indicates that the agent has chosen sales contract $(x, p) \in K_i$ from firm i, while $(i, x, p) = (0, 0, 0) \in \mathbb{K}$ indicates that the agent has chosen to abstain from contracting altogether. Note that the set $\mathbb{K}$ is a closed subset of the compact set $I \times X \times D$.[3] Thus, $\mathbb{K}$ is a compact set.

Example 1 *(Mutual Funds) Consider the market for mutual funds. Let $(Z, B(Z), \eta)$ be a probability space where Z is a metric space equipped with the Borel σ-field $B(Z)$, and η is a probability measure. Suppose that the set of all possible mutual funds is represented by a set X of $B(Z)$-measurable functions defined on the state space Z taking values in some closed bounded interval, $[L, H]$, where $L \leq 0 < H$.[4] In this example, each fund is represented by a function $f = f(\cdot) \in X$ summarizing the state-contingent returns the fund is capable of generating. Finally, assume that the set of state-contingent mutual fund return functions, X, has the following properties:*

(i) X contains a fund,

$$0(\cdot) : Z \to [L, H],$$

such that

$$0(z) = 0 \text{ for all } z \in Z.$$

(ii) X is sequentially compact for the topology of pointwise convergence on Z, that is, for any sequence of funds $\{f^n\}_n$ in X there is a subsequence $\{f^{n_k}\}_k$ in X and a fund $\overline{f} \in X$ such that

$$f^{n_k}(z) \to \overline{f}(z) \text{ for all } z \in Z.$$

(iii) X contains no redundant funds, that is, if the returns to funds f and $\overline{f}$ in X differ in some state of nature $z' \in Z$, then the returns to funds f and $\overline{f}$ differ on a set of states having positive probability. Stated formally, X contains no redundant funds if for any pair of funds f and $\overline{f}$ in X

$$f(z') \neq \overline{f}(z') \text{ for some } z' \in Z, \text{ implies that}$$
$$\eta\left\{z \in Z : f(z) \neq \overline{f}(z)\right\} > 0.$$

[3] Equip I with the discrete metric $d_I(\cdot, \cdot)$ given by

$$d_I(i, i') = \begin{cases} 1 & \text{if } i \neq i' \\ 0 & \text{otherwise.} \end{cases}$$

[4] See Aliprantis and Border (1999) for a definition of $\Im$-measurability.

The uniform boundedness of X (by L and H) together with conditions (ii) and (iii) imply that the set of funds X is compact and metrizable for the topology of pointwise convergence on Z (see Proposition 1 in Tulcea (1973)). In particular, under (ii), (iii), and uniform boundedness,

$$d_\eta(f, \overline{f}) := \int_Z \left| f(z) - \overline{f}(z) \right| d\eta(z),$$

defines a metric on X which generates the topology of pointwise convergence[5] *and makes X a compact metric space. Taking as the "no contracting" fund the function* $0(\cdot)$*, the product set X satisfies (A-2)(i).*

2.1.2 Catalogs of Contracts

For each firm $i = 1, 2, \ldots, m$, let C_i be a nonempty, closed subset of K_i. We can think of the subset C_i as representing a *catalog* of contracts that the i^{th} firm might offer to the agent. For $i = 0, 1, 2, \ldots, m$, let $P_f(K_i)$ denote the collection of all possible catalogs, that is, the collection of *all nonempty, closed subsets of K_i*.[6] Since $K_i := X_i \times D$ is a compact metric space, the collection of catalogs, $P_f(K_i)$, equipped with the Hausdorff metric h is automatically compact (see Aliprantis and Border (1999) for the definition of the Hausdorff metric and a discussion).

If firms compete via catalogs, then their strategy choices can be summarized via a catalog profile,

$$(C_1, \ldots, C_m) \,. \tag{2}$$

Here, the i^{th} component of the m-tuple $(C_1, \ldots, C_m)$ is the catalog offered by the i^{th} firm to the agent. Let

$$\mathbf{P} := P_f(K_1) \times \cdots \times P_f(K_m)$$

denote the space of all catalog profiles. If $\mathbf{P}$ is equipped with the metric $h_{\mathbf{P}}$ given by

$$h_{\mathbf{P}} \left((C_1, \ldots, C_m), (C'_1, \ldots, C'_m) \right) := \max\{h(C_i, C'_i) : i = 1, 2, \ldots, m\}, \tag{3}$$

then the space of catalog profiles $(\mathbf{P}, h_{\mathbf{P}})$ is a compact metric space.

2.1.3 The Agent's Choice Problem Under Catalog Competition

We shall assume that

(A-3) the agent's utility function,

$$v(\cdot, \cdot, \cdot, \cdot) : T \times I \times X \times D \to R, \tag{4}$$

[5] Thus, for any sequence of funds $\{f^n\}_n \subseteq X$ and fund $\overline{f} \in X$, $d_\eta(f^n, \overline{f}) \to 0$ if and only if

$$f^n(z) \to \overline{f}(z) \text{ for all } z \in Z.$$

[6] Note that since $K_0 = \{(0,0)\}$, $P_f(K_0)$ consists of one nonempty, closed subset, namely the set $\{(0,0)\}$.

is such that, (i) for each $t \in T$, $v(t, \cdot, \cdot, \cdot)$ is continuous and for each $(i, x, p) \in I \times X \times D$, $v(\cdot, i, x, p)$ is $\Im$-measurable, (ii) $v(t, i, 0, 0) \leq v(t, 0, 0, 0)$ for all $t \in T$ and $i = 1, 2, \ldots, m$, and (iii) for each $(t, i, x) \in T \times I \times X$, $v(t, i, x, \cdot)$ is strictly decreasing on D.

Note that we allow the agent's utility to depend not only on the sales contract (x, p) but also on brand name i (i.e., the name of the firm with which the agent contracts). However, by (A-3)(ii) if the agent is to derive any utility from a firm's brand name beyond the reservation level, $v(t, 0, 0, 0)$, then the agent must enter into a contract with the firm. Allowing utility to depend on brand names *does not* rule out the possibility that some (or all) types of the agent are completely indifferent to brand names.

Example 2 *(The Agent's Utility Function for the Mutual Fund Example)*
As in example 1, let X be a compact metric space of state-contingent return functions representing mutual funds. Suppose that the agent has conditional probability beliefs over the states-of-nature given by $\zeta(\cdot|\cdot)$ where for each closed set $E \subset Z$, the function

$$\zeta(E|\cdot) : T \to [0, 1]$$

is continuous. Let

$$u(\cdot, \cdot, \cdot) : T \times I \times [L, H] \to R$$

be a function such that (i) for each $t \in T$, $u(t, \cdot, \cdot)$ is continuous on $I \times [L, H]$, (ii) for each $(i, c) \in I \times [L, H]$, $u(\cdot, i, c)$ is $\Im$-measurable, and (iii) for each $i \in 1, 2, \ldots, m$, $u(t, i, 0) \leq u(t, 0, 0)$ on T. Finally, let the type t agent's (expected) utility over company-fund pairs be given by

$$v(t, i, f, p) := \int_Z u(t, i, f(z))\zeta(dz|t) - p,$$

where $p \in D$ is the up front cost of purchasing fund f. Specified in this way, the agent's utility function satisfies assumptions (A-3).

Given catalog profile,

$$(C_1, \ldots, C_m),$$

the agent's choice set is given by

$$\Gamma(C_1, \ldots, C_m) := \{(i, x, p) \in \mathbb{K} : (x, p) \in C_i\}, \tag{5}$$

and the agent's choice problem is given by

$$\max \{v(t, i, x, p) : (i, x, p) \in \Gamma(C_1, \ldots, C_m)\}. \tag{6}$$

Under assumptions (A-1)-(A-3), for each t the agent's choice problem has a solution. Let

$$v^*(t, C_1, \ldots, C_m) := \max \{v(t, i, x, p) : (i, x, p) \in \Gamma(C_1, \ldots, C_m)\} \tag{7}$$

and

$$\Phi(t, C_1, \ldots, C_m) := \{(i, x, p) \in \Gamma(C_1, \ldots, C_m) : v(t, i, x, p) = v^*(t, C_1, \ldots, C_m)\}. \quad (8)$$

The set-valued mapping

$$(C_1, \ldots, C_m) \to \Phi(t, C_1, \ldots, C_m)$$

is a type t agent's best response mapping. For each catalog profile

$$(C_1, \ldots, C_m) \in P_f(K_1) \times \cdots \times P_f(K_m),$$

$$\Phi(t, C_1, \ldots, C_m) \text{ is a nonempty closed subset of } \mathbb{K}.$$

The following Proposition summarizes the continuity and measurability properties of the mappings, Γ and Φ, and the optimal utility function, v^*.[7]

Proposition *(Continuity and measurability properties): Suppose assumptions (A-1)-(A-3) hold. Then the following statements are true. (a) The choice correspondence* $\Gamma(\cdot, \ldots, \cdot)$ *is* $h_{\mathbf{P}}$*-continuous on the space of catalog profiles* $\mathbf{P}$ *(i.e., is continuous with respect to the metric* $h_{\mathbf{P}}$*), (b) The function* $v^*(\cdot, \cdot, \ldots, \cdot)$ *is* $h_{\mathbf{P}}$*-continuous on* $\mathbf{P}$ *for each* $t \in T$*, and is* $\Im$*-measurable on* T *for each* $(C_1, \ldots, C_m) \in \mathbf{P}$*. (c) For each* $t \in T$*,* $\Phi(t, \cdot, \ldots, \cdot)$ *is* $h_{\mathbf{P}}$ *-upper semicontinuous on* $\mathbf{P}$ *and* $\Phi(\cdot, \cdot, \ldots, \cdot)$ *is* $\Im \times B(\mathbf{P})$ *-measurable on* $T \times \mathbf{P}$.[8]

The proof of the Proposition above follows from Propositions 4.1 and 4.2 in .Page (1992).

2.1.4 Contracting Mechanisms

We shall denote by $\Sigma(C_1, \ldots, C_m)$ the set of all $\Im$-measurable selections from the best response mapping,

$$t \to \Phi(t, C_1, \ldots, C_m),$$

that is, the set of $\Im$-measurable functions

$$t \to (i(t), x(t), p(t))$$

such that

$$(i(t), x(t), p(t)) \in \Phi(t, C_1, \ldots, C_m) \text{ for all } t \in T.$$

[7]For the continuity and measurability properties of the optimal utility function v^* for the case where $v(t, \cdot, \cdot, \cdot)$ is only upper semicontinuous on $I \times X \times D$ for each $t \in T$, see Balder and Yannelis (1993) and Page (1992).

[8]Here $B(\mathbf{P})$ denotes the Borel σ-field in the compact metric space $(\mathbf{P}, h_{\mathbf{P}})$. Moreover,

$$B(\mathbf{P}) = B(P_f(K_1)) \times \cdots \times B(P_f(K_m)),$$

where $B(P_f(K_j))$ denotes the Borel σ-field in the compact metric space of catalogs $(P_f(K_j), h)$ (see Aliprantis and Border (1999) Theorem 4.43, p. 146).

We shall refer to all $\Im$-measurable functions $(i(\cdot), x(\cdot), p(\cdot))$ as competitive contracting mechanisms and we shall refer to all competitive contracting mechanisms contained in $\Sigma(C_1, \ldots, C_m)$ as *viable mechanisms*. Under viable mechanism

$$(i(\cdot), x(\cdot), p(\cdot)) \in \Sigma(C_1, \ldots, C_m),$$

it is intended that a type $t \in T$ agent enter into contract $(x(t), p(t))$ with firm $i(t)$. Moreover, because $(i(t), x(t), p(t)) \in \Phi(t, C_1, \ldots, C_m)$ it is reasonable to assume that a type t agent will do as intended by the mechanism (i.e., will obey the mechanism).

By the Kuratowski - Ryll-Nardzewski Selection Theorem (see Aliprantis and Border (1999), p. 567), for any catalog profile, $(C_1, \ldots, C_m) \in \mathbf{P}$, the set of viable mechanisms, $\Sigma(C_1, \ldots, C_m)$, is nonempty. Thus, mechanism $(i(\cdot), x(\cdot), p(\cdot))$ is viable if and only if

$$v(t, i(t), x(t), p(t)) = \max \{v(t, i, x, p) : (i, x, p) \in \Gamma(C_1, \ldots, C_m)\} \text{ for all } t \in T.$$

By the *Delegation Principle* (Page (1992, 1999), Page and Monteiro (2003)), a competitive contracting mechanism $(i(\cdot), x(\cdot), p(\cdot))$ is rational and incentive compatible if and only if $(i(\cdot), x(\cdot), p(\cdot))$ is contained in $\Sigma(C_1, \ldots, C_m)$ for some catalog profile $(C_1, \ldots, C_m) \in \mathbf{P}$. Recall that a competitive contracting mechanism $(i(\cdot), x(\cdot), p(\cdot))$ is rational if for all agent types t in T

$$v(t, i(t), x(t), p(t)) \geq v(t, 0, 0, 0), \tag{9}$$

and it is incentive compatible if for all agent types t and t' in T

$$v(t, i(t), x(t), p(t)) \geq v(t, i(t'), x(t'), p(t')). \tag{10}$$

By the Delegation Principle, a competitive contracting mechanism is rational and incentive compatible if and only if it is viable.[9]

2.2 Firms

2.2.1 The Firm's Profit Function

We shall assume that

(A-4) the *jth* firm's profit is given by the function,

$$\pi_j(\cdot, \cdot, \cdot, \cdot) : T \times I \times X \times D \to R,$$

[9] In a competitive environment, who or what in the economy chooses the contracting mechanism? Such a mechanism would seem to require that firms choose a mediator who in turn chooses the mechanism - or at least, that firms act cooperatively in choosing the mechanism. This is precisely where the delegation principle comes into play. It follows from the delegation principle that the choice of a competitively viable direct contracting mechanisms can be *decentralized*. In particular, rather than have the agent report his type to a centralized contracting mechanism (however such a mechanism is chosen), instead each firm can simply offer the agent a catalog of contracts (i.e., a set of contracts *not* indexed by agent types) from which to choose. Given the catalog profile offered by firms, a viable mechanism will then emerge from the optimizing behavior of the agent for each of his types.

where (i) for each $t \in T$, $\pi_j(t, \cdot, \cdot, \cdot)$ is continuous and for each $(i, x, p) \in I \times X \times D$, $\pi_j(\cdot, i, x, p)$ is $\Im$-measurable, and (ii) there exists a μ-integrable function $\xi_j(\cdot) : T \to R$ such that for each for each $(i, x, p) \in I \times X \times D$, $|\pi_j(t, i, x, p)| \leq \xi_j(t)$ for all $t \in T$.

Example 3 *(Firm Profit Functions for the Mutual Fund Example)*

As in example 1, let X be a compact metric space of state-contingent return functions representing mutual funds. For $j = 1, 2, \ldots, m$, let firm j's profit be given by

$$\pi_j(t, i, f, p) = (p - c_j(t, f))I_j(i)$$

where

$$I_j(i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

and where $c_j(\cdot, \cdot) : T \times X \to R_+$ is firm j's cost function, $\Im$-measurable in t and continuous in f. Here, $c_j(t, f)$ is the cost to company j of providing and maintaining fund $f \in X$ for a type t buyer. Specified in this way company profit functions satisfy assumptions (A-4).

2.2.2 Market Situations: Catalogs, Viable Mechanisms, and Profits

Each firm's expected profit is determined by the catalogs chosen by all firms *as well as* by the contracting mechanism which emerges as a result of the optimizing behavior of the agent for each of his types (i.e., optimizing behavior of buyers in the market). Thus, given catalogs $(C_1, \ldots, C_m) \in \mathbf{P}$ chosen by firms, if contracting mechanism $(i(\cdot), x(\cdot), p(\cdot)) \in \Sigma(C_1, \ldots, C_m)$ prevails, then the *jth* firm's expected profit is

$$\Pi_j(i(\cdot), x(\cdot), p(\cdot)) = \int_T \pi_j(t, i(t), x(t), p(t))d\mu(t). \tag{11}$$

Note that each firm j can abstain from participating by offering catalog $C_j = \{(0, 0)\}$.

Definition 1 *(Market Situations) Consider the profit-catalog profile pair (π, C), where $\pi = (\pi_1, \ldots, \pi_m) \in R^m$ and $C = (C_1, \ldots, C_m) \in \mathbf{P}$. We shall refer to the pair (π, C) as a* market situation *if there exists a viable contracting mechanism*

$$(i(\cdot), x(\cdot), p(\cdot)) \in \Sigma(C)$$

such that

$$\pi = \Pi(i(\cdot), x(\cdot), p(\cdot)) := (\Pi_1(i(\cdot), x(\cdot), p(\cdot)), \ldots, \Pi_m(i(\cdot), x(\cdot), p(\cdot)))\,.$$

We shall denote by

$$\mathbb{M} := \{(\pi, C) \in R^m \times \mathbf{P} : \pi = \Pi(i(\cdot), x(\cdot), p(\cdot)) \text{ for some } (i(\cdot), x(\cdot), p(\cdot)) \in \Sigma(C)\}, \tag{12}$$

the set of all market situations.

Theorem 1 *(Compactness of the set of market situations)*

Suppose assumptions (A-1)-(A-4) hold. Then the set of market situations $\mathbb{M}$ is a nonempty, compact subset of $R^m \times \mathbf{P}$.

Proof. The boundedness of $\mathbb{M}$ follows immediately from the compactness of the space of catalog profiles $\mathbf{P}$, and the integrability assumption (A-4)(ii). To demonstrate closedness, let $\{(\pi_1^k, \ldots, \pi_m^k, C_1^k, \ldots, C_m^k)\}_k$ be a sequence of market situations in $\mathbb{M}$ converging to $(\bar{\pi}_1, \ldots, \bar{\pi}_m, \bar{C}_1, \ldots, \bar{C}_m) \in R^m \times \mathbf{P}$. We must show that

$$(\bar{\pi}_1, \ldots, \bar{\pi}_m, \bar{C}_1, \ldots, \bar{C}_m) \in \mathbb{M}.$$

The proof will be complete if we can show that there is a mechanism $(\bar{\imath}(\cdot), \bar{x}(\cdot), \bar{p}(\cdot)) \in \Sigma(\bar{C}_1, \ldots, \bar{C}_m)$ such that

$$\bar{\pi}_j = \Pi_j(\bar{\imath}(\cdot), \bar{x}(\cdot), \bar{p}(\cdot)) \text{ for all } j.$$

Let $\{(i^k(\cdot), x^k(\cdot), p^k(\cdot))\}_k$ be a sequence of mechanisms such that for all k

$$(i^k(\cdot), x^k(\cdot), p^k(\cdot)) \in \Sigma(C_1^k, \ldots, C_m^k),$$
$$\text{and}$$
$$\pi_j^k = \Pi_j(i^k(\cdot), x^k(\cdot), p^k(\cdot)) \text{ for all } j.$$

Consider the $\Im$-measurable set-valued mapping given by

$$t \to Ls\left\{\pi(t, i^k(t), x^k(t), p^k(t))\right\},$$

where

$$\pi(t, i^k(t), x^k(t), p^k(t)) = \Big(\pi_1(t, i^k(t), x^k(t), p^k(t)), \ldots, \pi_m(t, i^k(t), x^k(t), p^k(t))\Big).$$

This mapping has nonempty, compact values in R^m for all t. By Fatou's lemma in several dimensions (Artstein (1979)), there exits a

$$\mu\text{-integrable function } \bar{\pi}(\cdot) : T \to R^m \text{ such that}$$

$$\bar{\pi}(t) \in Ls\left\{\pi(t, i^k(t), x^k(t), p^k(t))\right\} \text{ for } t \in T \backslash N,\ \mu(N) = 0, \text{ and}$$

$$\bar{\pi} = \textstyle\int_T \bar{\pi}(t) d\mu(t).$$

Next, consider the minimization problem

$$\min\left\{\|\bar{\pi}(t) - \pi(t, i, x, p)\| : (i, x, p) \in \Phi(t, \bar{C}_1, \ldots, \bar{C}_m)\right\},$$

and let

$$\delta(t) := \min\left\{\|\bar{\pi}(t) - \pi(t, i, x, p)\| : (i, x, p) \in \Phi(t, \bar{C}_1, \ldots, \bar{C}_m)\right\} \text{ for all } t \in T.$$

By Schal (1974), there exists a mechanism $(\bar{\imath}(\cdot), \bar{x}(\cdot), \bar{p}(\cdot)) \in \Sigma(\bar{C}_1, \ldots, \bar{C}_m)$ such that

$$\|\bar{\pi}(t) - \pi(t, \bar{\imath}(t), \bar{x}(t), \bar{p}(t))\| = \delta(t) \text{ for all } t \in T.$$

We will show that $\delta(t) = 0$ for all $t \in T \backslash N$.

First, let $t \in T \backslash N$ be given and without loss of generality assume that

$$\pi(t, i^k(t), x^k(t), p^k(t)) \to \bar{\pi}(t)$$

By the compactness of $I \times X \times D$, there is a subsequence $\left\{(i^{k_v}(t), x^{k_v}(t), p^{k_v}(t))\right\}_v$ with

$$(i^{k_v}(t), x^{k_v}(t), p^{k_v}(t)) \to (\bar{\imath}, \bar{x}, \bar{p}) \in I \times X \times D$$

Because

$$(i^{k_v}(t), x^{k_v}(t), p^{k_v}(t)) \in \Phi(t, C_1^{k_v}, \ldots, C_m^{k_v}) \text{ for all } v$$
$$\text{and}$$
$$(C_1^{k_v}, \ldots, C_m^{k_v}) \to (\bar{C}_1, \ldots, \bar{C}_m) \in \mathbf{P},$$

by the upper semicontinuity of $\Phi(t, \cdot, \ldots, \cdot)$, we have $(\bar{\imath}, \bar{x}, \bar{p}) \in \Phi(t, \bar{C}_1, \ldots, \bar{C}_m)$. By the continuity of $\pi(t, \cdot, \cdot, \cdot)$, we have

$$\pi(t, i^{k_v}(t), x^{k_v}(t), p^{k_v}(t)) \to \bar{\pi}(t).$$

Thus, for this $t \in T \backslash N$, we have

$$\delta(t) = \|\bar{\pi}(t) - \pi(t, \bar{\imath}(t), \bar{x}(t), \bar{p}(t))\| = \|\bar{\pi}(t) - \pi(t, \bar{\imath}, \bar{x}, \bar{p})\| = 0.$$

We can conclude, therefore, that $(\bar{\imath}(\cdot), \bar{x}(\cdot), \bar{p}(\cdot)) \in \Sigma(\bar{C}_1, \ldots, \bar{C}_m)$ is such that

$$\pi(t, \bar{\imath}(t), \bar{x}(t), \bar{p}(t)) = \bar{\pi}(t) \text{ for all } t \in T \backslash N.$$

Thus,

$$\bar{\pi}_j = \Pi_j(\bar{\imath}(\cdot), \bar{x}(\cdot), \bar{p}(\cdot)) \text{ for all } j.$$

■

3 Farsighted Catalog Competition and Market Situation Games

Consider two market situations, (π^0, C^0) and (π^1, C^1), such that $\pi_j^1 > \pi_j^0$ for firms $j \in S$, S a nonempty subset of $N := \{1, 2, \ldots, m\}$. From the perspective of firms $j \in S$, market situation (π^1, C^1) is preferred to market situation (π^0, C^0). Three questions now arise: (i) Is it within the power of firms $j \in S$ acting collusively or acting independently but simultaneously to change the market situation from (π^0, C^0) to (π^1, C^1) by changing *their* catalogs? (ii) Will such a change trigger further catalog changes, and thus further profit outcome changes, possibly leaving some or all firms $j \in S$ not better off and possibly worse off? (iii) Is there a market situation which is stable in the sense that no firm or subset of firms has an incentive to change their catalog for fear that such a change might trigger a sequence of changes which makes the initially deviating firm or firms not better off and possibly worse off? These are the questions we now address.

3.1 Farsighted Dominance and Consistency

We begin with some definitions. Throughout we shall denote by S a *nonempty* subset of $N := \{1, 2, \ldots, m\}$.

Definition 2 *(Credible Change and Improvement) Let (π^0, C^0) and (π^1, C^1) be two market situations (i.e., pairs contained in $\mathbb{M}$), and let $S \subseteq N$.*

(1) (Credibly Change) *We say that firms $j \in S$ can credibly change the market situation from (π^0, C^0) to (π^1, C^1), denoted*

$$(\pi^0, C^0) \rightarrow_S (\pi^1, C^1),$$

*if $C_j^0 = C_j^1$ for all firms $j \in N \backslash S$ (i.e, j*not *contained in S).*

(2) (Improvement) *We say that market situation (π^1, C^1) is an improvement over market situation (π^0, C^0) for firms $j \in S$, denoted*

$$(\pi^1, C^1) \succ_S (\pi^0, C^0),$$

if $\pi_j^1 > \pi_j^0$ for firms $j \in S$.

(3) (Farsighted Dominance) *We say that market situation (π, C) is farsightedly* dominated *by market situation (π^*, C^*), denoted*

$$(\pi^*, C^*) \rhd\rhd (\pi, C),$$

if there exists a finite sequence of market situations, $(\pi^0, C^0), \ldots, (\pi^n, C^n)$, and a corresponding sequence of sets of firms, $S^1, \ldots, S^n$, such that

$$\begin{gathered}(\pi, C) := (\pi^0, C^0) \text{ and } (\pi^*, C^*) := (\pi^n, C^n), \text{ and} \\ \text{for } k = 1, 2, \ldots, n, \\ (\pi^{k-1}, C^{k-1}) \rightarrow_{S^k} (\pi^k, C^k) \text{ and} \\ (\pi^n, C^n) \succ_{S^k} (\pi^{k-1}, C^{k-1}).\end{gathered}$$

Thus, market situation (π, C) is farsightedly dominated by market situation (π^*, C^*) if (i) there is a finite sequence of credible changes in market situations starting with situation (π, C) and ending with situation (π^*, C^*), and if (ii) the profit outcome π^* in ending market situation (π^*, C^*) is such that for each k and each firm $j \in S^k$, *profit in the ending situation* is greater than profit in the situation (π^{k-1}, C^{k-1}) that firms $j \in S^k$ changed - that is, $\pi_j^* := \pi_j^n > \pi_j^{k-1}$ for each firm $j \in S^k$.

3.1.1 Farsightedly Consistent Market Situations

Again we begin with definitions.

Definition 3 *(Farsighted Consistency) A subset $\mathbb{F}$ of market situations is said to be farsightedly consistent if for each market situation $(\pi^0, C^0) \in \mathbb{F}$ the following is true: given any $(\pi^1, C^1) \in \mathbb{M}$ such that*

$$(\pi^0, C^0) \rightarrow_S (\pi^1, C^1) \text{ for firms } S \subseteq N,$$

there exists another market situation $(\pi^2, C^2) \in \mathbb{F}$ with

$$\text{either } (\pi^2, C^2) = (\pi^1, C^1) \text{ or } (\pi^2, C^2) \rhd\rhd (\pi^1, C^1)$$

such that,

$$(\pi^2, C^2) \not\succ_S (\pi^0, C^0).$$

A subset $\mathbb{F}^*$ of market situations is said to be the largest farsightedly consistent set if for any farsightedly consistent set $\mathbb{F}$ it is true that $\mathbb{F} \subseteq \mathbb{F}^*$.

In words, a set $\mathbb{F}$ of market situations is farsightedly consistent, if given any market situation (π^0, C^0) in $\mathbb{F}$ and any credible S-deviation to market situation $(\pi^1, C^1) \in \mathbb{M}$, there exists another market situation (π^2, C^2) in $\mathbb{F}$ such that *either* $(\pi^2, C^2) = (\pi^1, C^1)$ *or* (π^2, C^2) farsightedly dominates (π^1, C^1) and such that (π^2, C^2) is *not* an S-improvement over (π^0, C^0). Thus, $\mathbb{F}$ is farsightedly consistent if given any market situation (π^0, C^0) in $\mathbb{F}$, any credible S -deviation to a market situation (π^1, C^1) in $\mathbb{M}$ might lead to an ending market situation (π^2, C^2) in $\mathbb{F}$ which makes all or some firms in S not better off and possibly worse off.

3.2 Market Situation Games

We may think of the set of market situations $\mathbb{M}$ equipped with binary relation $\rhd\rhd$ as describing an abstract market situation game in the sense of von Neumann-Morgenstern. We say that a market situation $(\pi^*, C^*) \in \mathbb{M}$ is farsightedly consistent (i.e., is an equilibrium of the game $(\mathbb{M}, \rhd\rhd)$) if (π^*, C^*) is contained in the largest farsightedly consistent set, that is, if

$$(\pi^*, C^*) \in \mathbb{F}^*.$$

Chwe (1994) has shown that for all games, such as the market situation game $(\mathbb{M}, \rhd\rhd)$, there exists a unique, largest farsightedly consistent set (see Chwe (1994), Proposition 1). However, like the core, the largest farsightedly consistent set $\mathbb{F}^*$ may be empty. What guarantees that $\mathbb{F}^* \neq \emptyset$?

3.2.1 Finite Market Situation Subgames and Approximation

We shall refer to any market situation game $(\mathbb{D}, \rhd\rhd)$ where $\mathbb{D} \subseteq \mathbb{M}$ is a *finite* subset of $\mathbb{M}$ as a *finite market situation subgame.* By Theorem 1, the set of market situations $\mathbb{M}$ is compact. This natural compactness in the space of market situations has important implications for the relationship between the market situation game $(\mathbb{M}, \rhd\rhd)$ and its finite subgames. In particular, given the compactness of $\mathbb{M}$, there exists a finite subgame “arbitrarily close” to the market situation game $(\mathbb{M}, \rhd\rhd)$. By “arbitrarily close,” we mean the following: Let $d_{\mathbb{M}}(\cdot,\cdot)$ be the metric on $\mathbb{M}$ given by

$$d_{\mathbb{M}}((\pi, C),(\pi', C')) := \max\{d_{R^m}(\pi,\pi'), h_{\mathbf{P}}(C, C')\}.$$

Here d_{R^m} denotes the standard Euclidean metric on R^m. Since $(\mathbb{M}, d_{\mathbb{M}})$ is compact, for every $\varepsilon > 0$, there exists a finite subset of market situations $\mathbb{D}_\varepsilon$ such that

$$\mathbb{M} = \cup_{(\pi',C')\in\mathbb{D}_\varepsilon}\{(\pi, C) \in \mathbb{M} : d_{\mathbb{M}}((\pi, C),(\pi', C')) < \varepsilon\}.$$

Thus, each market situation in the game $(\mathbb{M}, \rhd\rhd)$ is within ε of some market situation in the finite market situation subgame $(\mathbb{D}_\varepsilon, \rhd\rhd)$. More importantly, *all* such finite market situation subgames have nonempty farsightedly consistent sets.

Theorem 2 *(Nonemptiness of $\mathbb{F}^*_\varepsilon$ for finite subgames $(\mathbb{D}_\varepsilon, \rhd\rhd)$)*
For any $\varepsilon > 0$ the finite market situation subgame $(\mathbb{D}_\varepsilon, \rhd\rhd)$ corresponding to the market situation game $(\mathbb{M}, \rhd\rhd)$ has a nonempty largest farsightedly consistent set $\mathbb{F}^*_\varepsilon$. Moreover, $\mathbb{F}^*_\varepsilon$ is externally $\rhd\rhd$-stable, that is, for all $(\pi, C) \in \mathbb{D}_\varepsilon\backslash\mathbb{F}^*_\varepsilon$, there exists $(\pi^*, C^*) \in \mathbb{F}^*_\varepsilon$, such that $(\pi^*, C^*) \rhd\rhd (\pi, C)$.

Proof. First note that for all $S \subseteq N$, the relation $\succ_S$ defined on $\mathbb{M}$ is irreflexive (i.e., $(\pi, C) \not\succ_S (\pi, C)$ for $(\pi, C) \in \mathbb{M}$). Thus, the proof follows immediately from the Corollary to Proposition 2 in Chwe (1994). ■

As a corollary, we conclude that if the set of market situations is finite, then the game has a unique, nonempty largest consistent set. The set of market situations $\mathbb{M}$ will be finite if, for example, the set X of potential products and the set D of potential prices are finite.

By Proposition 3 in Chwe (1994), if the market situation in the game $(\mathbb{M}, \rhd\rhd)$ has a $\rhd\rhd$-stable set, for $\mathbb{M}$ finite or infinite, then the game has a unique, nonempty largest consistent set. A set $\mathbb{S}^* \subseteq \mathbb{M}$ is a $\rhd\rhd$-stable set if it is both externally *and* internally $\rhd\rhd$-stable.[10]

3.3 Market Situation Games with Respect to Path Dominance

We say that a sequence of market situations $\{(\pi^k, C^k)\}_k$ forms a *farsighted domination path* through $\mathbb{M}$ if for any two consecutive market situations (π^{k-1}, C^{k-1}) and (π^k, C^k), (π^{k-1}, C^{k-1}) is farsightedly dominated by (π^k, C^k), that is, if for any two consecutive market situations (π^{k-1}, C^{k-1}) and (π^k, C^k), $(\pi^{k-1}, C^{k-1}) \lhd\lhd (\pi^k, C^k)$.

We say that market situation (π^0, C^0) is farsightedly path dominated by market situation (π^1, C^1) - or that (π^1, C^1) is *reachable* from (π^0, C^0) - if there is a *finite* farsighted domination path starting at (π^0, C^0) and ending at (π^1, C^1).

[10] $\mathbb{S}^*$ is externally $\rhd\rhd$-stable if for all $(\pi^0, C^0) \in \mathbb{M}\backslash\mathbb{S}^*$, there exists $(\pi^1, C^1) \in \mathbb{S}^*$, such that $(\pi^1, C^1) \rhd\rhd (\pi^0, C^0)$. $\mathbb{S}^*$ is internally $\rhd\rhd$-stable if for all (π^0, C^0) and (π^1, C^1) contained in $\mathbb{S}^*$, neither $(\pi^1, C^1) \rhd\rhd (\pi^0, C^0)$ nor $(\pi^0, C^0) \rhd\rhd (\pi^1, C^1)$.

We can use the notion of reachability to define a new relation on the set of market situations $\mathbb{M}$. In particular, for any two market situations (π^0, C^0) and (π^1, C^1) define

$$(\pi^1, C^1) \trianglerighteq (\pi^0, C^0) \text{ if and only if } \begin{cases} (\pi^1, C^1) \text{ is reachable from } (\pi^0, C^0), \text{ or} \\ (\pi^1, C^1) = (\pi^0, C^0). \end{cases} \quad (13)$$

The relation $\trianglerighteq$ is a weak ordering on the set of market situations $\mathbb{M}$. In particular, $\trianglerighteq$ is reflexive ($(\pi, C) \trianglerighteq (\pi, C)$) and $\trianglerighteq$ is transitive ($(\pi^2, C^2) \trianglerighteq (\pi^1, C^1)$ and $(\pi^1, C^1) \trianglerighteq (\pi^0, C^0)$ implies that $(\pi^2, C^2) \trianglerighteq (\pi^0, C^0)$). We shall refer to the relation $\trianglerighteq$ as the *path dominance relation* (i.e., the PD relation).[11]

A market situation game with respect to path dominance (i.e., a PD market situation game) is specified by the pair $(\mathbb{M}, \trianglerighteq)$.

3.3.1 $\trianglerighteq$-Stable Market Situations: Two Definitions and Two Results

We begin by defining the notions of $\trianglerighteq$-stability and inductivity.

Definition 4 *($\trianglerighteq$-Stable Sets)*

Let $(\mathbb{M}, \trianglerighteq)$ be an PD market situation game. A subset $\mathbb{V}$ of market situations in $\mathbb{M}$ is $\trianglerighteq$-stable for $(\mathbb{M}, \trianglerighteq)$ if

(a) (internal $\trianglerighteq$-stability) for all (π^0, C^0) and (π^1, C^1) in $\mathbb{V}$, with $(\pi^0, C^0) \neq (\pi^1, C^1)$, neither $(\pi^1, C^1) \trianglerighteq (\pi^0, C^0)$ nor $(\pi^0, C^0) \trianglerighteq (\pi^1, C^1)$, and

(b) (external $\trianglerighteq$-stability) for all $(\pi^0, C^0) \notin \mathbb{V}$, there exists $(\pi^1, C^1) \in \mathbb{V}$ such that $(\pi^1, C^1) \trianglerighteq (\pi^0, C^0)$.

In other words, a nonempty subset of market situations $\mathbb{V}$ is $\trianglerighteq$-stable for $(\mathbb{M}, \trianglerighteq)$ if (π^0, C^0) and (π^1, C^1) are in $\mathbb{V}$, $(\pi^0, C^0) \neq (\pi^1, C^1)$, then (π^1, C^1) is *not* reachable from (π^0, C^0), nor is (π^0, C^0) reachable from (π^1, C^1), and if $(\pi^0, C^0) \notin \mathbb{V}$, then there exists $(\pi^1, C^1) \in \mathbb{V}$ reachable from (π^0, C^0).

Assuming that the set of market situations is finite, if $\mathbb{V}$ is a $\trianglerighteq$-stable set of market situations and if market situation (π^0, C^0) is in $\mathbb{V}$, then it follows from Theorem 3 in Page and Wooders (2004) that any credible S-deviation to market situation $(\pi^1, C^1) \in \mathbb{M}$, will either not make all the deviating firms better off or will trigger a finite sequence of deviations leading back to the original market situation (π^0, C^0) - thus making none of the initially deviating firms better off.

Definition 5 *(Inductive PD Market Situation Games)*

A PD market situation game $(\mathbb{M}, \trianglerighteq)$ is said to be inductive if given any farsighted domination path $\left\{(\pi^k, C^k)\right\}_k$ through $\mathbb{M}$ there exists a market situation $(\pi^*, C^*) \in \mathbb{M}$ such that

$$(\pi^*, C^*) \trianglerighteq (\pi^k, C^k) \text{ for all } k.$$

Market situation (π^*, C^*) above is referred to as the majorant of the sequence.

[11] The relation $\trianglerighteq$ is sometimes referred to as the transitive closure of the farsighted dominance relation, $\lhd\lhd$.

If the set of market situations is finite, then the market situation game is automatically inductive. To see this, let $\left\{(\pi^k, C^k)\right\}_k$ be a farsighted domination path through $\mathbb{M}$. If the sequence $\left\{(\pi^k, C^k)\right\}_k$ is finite, then the last market situation in the sequence is a majorant. If $\left\{(\pi^k, C^k)\right\}_k$ is infinite, then because the set of market situations is finite, the sequence contains at least one market situation which is repeated an infinite number of times, and this infinitely repeated market situation is a majorant.

The following results due to Berge (2001, Theorem 1, p. 14 and Theorem 2, p. 15) clarify the close relationship which exists between inductivity and $\trianglerighteq$-stability.

Theorem 3 *(The relationship between inductivity. and $\trianglerighteq$-stability for PD market situation games)*

(1) Every inductive PD market situation game possesses a $\trianglerighteq$-stable set.

(2) Every PD market situation game possessing a finite $\trianglerighteq$-stable set is inductive.

There is also a close relationship between $\rhd\rhd$-stability and inductivity. If the market situation game $(\mathbb{M}, \rhd\rhd)$ has a *finite* $\rhd\rhd$-stable set $\mathbb{S}^*$, then the corresponding PD market situation game $(\mathbb{M}, \trianglerighteq)$ is inductive. To see this let $\{(\pi^n, C^n)\}_n$ be any farsighted domination path through $\mathbb{M}$ and for each n let $\{(\pi^{*n}, C^{*n})\}_n$ be market situations contained in the $\rhd\rhd$-stable set $\mathbb{S}^*$ such that for all n, $(\pi^{*n}, C^{*n}) \rhd\rhd (\pi^n, C^n)$. Since $\mathbb{S}^*$ is finite there is at least one market situation, say (π^{*k}, C^{*k}), that appears in the sequence $\{(\pi^{*n}, C^{*n})\}_n$ an infinite number of times. Given that $(\pi^{*n}, C^{*n}) \rhd\rhd (\pi^n, C^n)$ implies that $(\pi^{*n}, C^{*n}) \trianglerighteq (\pi^n, C^n)$, it follows that

$$(\pi^{*k}, C^{*k}) \trianglerighteq (\pi^n, C^n) \text{ for all } n.$$

Our last result on farsighted stability, a straightforward consequence of Theorems 3 and 7 in Page and Wooders (2004), clarifies how the $\trianglerighteq$-stable sets of a PD market situation game are related to the largest consistent set for the corresponding market situation game.

Theorem 4 *(The relationship between $\trianglerighteq$-stable sets and the largest consistent set for a finite market situation game)*

Let $(\mathbb{M}, \trianglerighteq)$ be a finite PD market situation game and let $\mathbb{F}^*$ be the unique, nonempty largest farsightedly consistent set for the corresponding finite market situation game $(\mathbb{M}, \rhd\rhd)$. There exists a $\trianglerighteq$-stable set $\mathbb{V}$ for $(\mathbb{M}, \trianglerighteq)$ such that $\mathbb{V} \subseteq \mathbb{F}^*$.

Finite PD market situation games often possess several $\trianglerighteq$-stable sets. As the next example illustrates, there may exist $\trianglerighteq$-stable sets that are proper subsets of the largest consistent set, as well as stable sets that have no intersection with the largest consistent set.

Example 4 *Consider two firms which compete for the business of a single, privately informed agent. Suppose that there are only three contracts, $0 := (0,0)$, $f_A := (x_A, p_A)$, and $f_B := (x_B, p_B)$ and that*

$$\textit{for all agent types } t \in T, v(t, i, 0, 0) < v(t, 0, 0, 0) \textit{ for } i = 1, 2. \; (*)$$

The space of catalog profiles is given by

$$\mathbf{P} = P_f(K_1) \times P_f(K_2),$$
$$\text{where for } i = 1, 2,$$
$$P_f(K_i) = \{\{0\}, \{f_A\}, \{f_B\}, \{0, f_A\}, \{0, f_B\}, \{f_A, f_B\}, \{0, f_A, f_B\}\},$$

but given (*), we need only consider the catalog game played over the collection of catalogs given by

$$\{\{0\}, \{f_A\}, \{f_B\}, \{f_A, f_B\}\}.$$

Suppose now that the expected payoffs to firms for all possible catalog profiles are given by Table 1.

Firm 1 ↕ / Firm 2 ↔	$\{0\}$	$\{f_A\}$	$\{f_B\}$	$\{f_A, f_B\}$
$\{0\}$	(0,0) $_1$	(0,-2) $_2$	(0,5) $_3$	(0,1) $_4$
$\{f_A\}$	(3,0) $_5$	(1,-3) $_6$	(1,1) $_7$	(1,-1) $_8$
$\{f_B\}$	(0,0) $_9$	(-1,-2) $_{10}$	(0,1) $_{11}$	(-2,1) $_{12}$
$\{f_A, f_B\}$	(2,0) $_{13}$	(1,-3) $_{14}$	(0,-1) $_{15}$	(-1,-1) $_{16}$

Table 1: Catalog Profiles and Payoffs

For example, if firm 1 offers catalog $\{f_A, f_B\}$ while firm 2 offers catalog $\{f_A\}$, then the corresponding expected payoffs are given in cell 14 of the Table.[12] *In particular, under catalog profile $C = (C_1, C_2) = (\{f_A, f_B\}, \{f_A\})$,*

$$\Pi_1(\{f_A, f_B\}, \{f_A\}) = 1,$$
$$\text{and}$$
$$\Pi_2(\{f_A, f_B\}, \{f_A\}) = -3.$$

The market situation corresponding to the payoffs in cell 14 is given by $(\pi, C) = (1, -3, \{f_A, f_B\}, \{f_A\}) \in \mathbb{M}$.

The PD market situation game $(\mathbb{M}, \trianglerighteq)$ with payoffs given in Table 1 has three $\trianglerighteq$-stable sets,

$$\mathbb{V}_1 = \{(3, 0, \{f_A\}, \{0\})\},$$
$$\mathbb{V}_2 = \{(1, 1, \{f_A\}, \{f_B\})\},$$
$$\mathbb{V}_3 = \{(2, 0, \{f_A, f_B\}, \{0\})\}.$$

The corresponding market situation game $(\mathbb{M}, \triangleright\triangleright)$ has largest farsightedly consistent set given by

$$\mathbb{F}^* = \{(1, 1, \{f_A\}, \{f_B\}), (2, 0, \{f_A, f_B\}, \{0\})\}.$$

Thus, $\trianglerighteq$-stable sets $\mathbb{V}_2$ and $\mathbb{V}_3$ are proper *subsets of the largest farsightedly consistent set $\mathbb{F}^*$, while $\trianglerighteq$-stable set $\mathbb{V}_1$ has no intersection with $\mathbb{F}^*$.*[13]

[12] We have spared the reader the tedious details of computing the expected payoffs appearing in Table 1. For a similar example containing all the details see Page and Monteiro (2003).

[13] The farsightedly consistent set $\mathbb{F}^*$ in this example was computed using a *Mathematica* package developed by Kamat and Page (2001).

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